College of the Holy Cross, Fall Semester, 2018
MATH 351, Final Examination
Friday, December 14

Your Name: ______________________________

Instructions: Please show all work necessary to justify your answers. Use the back of the preceding page if you need more space for scratch work. There are 200 possible points distributed as below. The parts marked with (*) will be used in computing the Midterm 1 subscore.

Please do not write in the space below

Problem	Points/Poss
I	/ 30
II	/ 30
III	/ 30
IV	/ 40
V	/ 30
VI	/ 40
Midterm I Sub	/80
Total	/200

Have a peaceful and joyous holiday season!

I. Both parts of this question deal with $SL(2,\mathbb{Z})$, the set of 2×2 integer matrices of determinant 1, a group under matrix multiplication. Let

$$H = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \bmod 5 \right\}.$$

(A) (*) (15) Is H a subgroup of $SL(2,\mathbb{Z})$? Why or why not?

(B) (*) (15) Is the cyclic subgroup generated by

$$A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

in $SL(2, \mathbb{Z})$ finite or infinite? Explain.

II. Let $G = \langle a \rangle$ be a cyclic group of order 100.

(A) (*) (10) How many different generators does G have?

(B) (*) (10) What is the order of the element a^{30} in G?

(C) (*) (10) Suppose you know that a subgroup H of G contains both a^{30} and a^{56}. What can you say about the order of H?

III. (A) (10) Let $\alpha : G \to H$ be a group homomorphism. Show that $\alpha(G)$ is a subgroup of H.

(B) (20) *State and prove* the First Isomorphism Theorem for groups.

IV. All parts of this question refer to the group G of order 8 whose operation table is given below:

$\cdot$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_2	g_2	g_5	g_4	g_7	g_6	g_1	g_8	g_3
g_3	g_3	g_8	g_5	g_2	g_7	g_4	g_1	g_6
g_4	g_4	g_3	g_6	g_5	g_8	g_7	g_1	g_2
g_5	g_5	g_6	g_7	g_8	g_1	g_2	g_3	g_4
g_6	g_6	g_1	g_8	g_3	g_2	g_5	g_4	g_7
g_7	g_7	g_4	g_1	g_6	g_3	g_8	g_5	g_2
g_8	g_8	g_7	g_2	g_1	g_4	g_3	g_6	g_5

(A) (*) (5) What is the inverse of the element g_2?

(B) (*) (5) What elements are in the subgroup $\langle g_3 \rangle$?

(C) (*) (5) Is the subgroup $\langle g_3 \rangle$ normal in G? Why or why not?

Here is the group table of G again so you don't need to flip back to the last page:

$\cdot$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_2	g_2	g_5	g_4	g_7	g_6	g_1	g_8	g_3
g_3	g_3	g_8	g_5	g_2	g_7	g_4	g_1	g_6
g_4	g_4	g_3	g_6	g_5	g_8	g_7	g_1	g_2
g_5	g_5	g_6	g_7	g_8	g_1	g_2	g_3	g_4
g_6	g_6	g_1	g_8	g_3	g_2	g_5	g_4	g_7
g_7	g_7	g_4	g_1	g_6	g_3	g_8	g_5	g_2
g_8	g_8	g_7	g_2	g_1	g_4	g_3	g_6	g_5

(D) (*) (5) What is the center of G, that is, the subgroup $Z(G)$?

(E) (20) Construct the group table for the factor group $G/Z(G)$. To which "standard" group is this isomorphic?

V. (A) (15) Up to isomorphism, how many different *abelian* groups of order 600 are there? List one group from each isomorphism class.

(B) (15) Up to isomorphism, how many different groups of order 2018 are there? List one group from each isomorphism class. (Hint: This is *not* a long list!)

VI. (A) (20) Use the Sylow Theorems to show that there are no simple groups of order 100.

(B) (20) How many different Sylow 5-subgroups does the alternating group A_5 have?