College of the Holy Cross, Fall 2019
MATH 110-02 – Algebra Through History
Solutions for Final Exam, December 18, 2019

I. Answer any 8 of the following 12 short answer questions. If you submit answers for more than 8, only the best 8 will be used to compute your score.

A) (5) Give a *brief* description of the number system used by the Old Babylonian scribes (one sentence would suffice). Explain and/or give an example of one ambiguous feature of this system.

Answer: This was a positional base-60 system, but without a symbol for 0 and with no symbol indicating the start of the fractional part of the number. Either one of these features could produce ambiguities. For instance, an especially wide space between two base-60 digits might represent a zero or it might not. All numbers can be interpreted in different ways depending on where the fractional part begins.

B) (5) According to Jens Høyrup, what is the best way to describe the operations in the solution of the problem from the Old Babylonian tablet YBC 6967? (You don't need to reproduce the whole solution; just describe in general terms.)

Answer: According to Høyrup, the operations in the solution would be best described as "cut and paste" geometry on a rectangular figure with sides equal to the number x and its "reciprocal" $60/x$.

C) (5) Approximately when and where do we think Euclid was active? What evidence is there for this or any details of his life?

Answer: Euclid is thought to have been active around 300 BCE in Alexandria in Egypt. He probably moved there from Greece under the patronage of the first Ptolemy. The only evidence for this, though, comes from much later works such as the commentary on Book I of the *Elements* by the philosopher Proclus. Virtually no hard facts are known about his life at all.

D) (5) What effect did the discovery of *incommensurable magnitudes* apparently have on the presentation of basic mathematics in Euclid's *Elements*? Explain briefly.

Answer: The principal effect was apparently the separation of the discussion of geometric magnitudes and numbers into distinct books of the *Elements*. Magnitudes such as lengths and areas were never assigned numerical values by Euclid. In addition, numbers were strictly limited to positive whole numbers (and perhaps fractions of integers, if we interpret proportions that way).

E) (5) One translation of Proposition 2 from Book II of Euclid's *Elements* is as follows: "If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole." What algebraic equation is equivalent to this if a straight line of length x is cut into segments of lengths y, z, w?

Answer: The statement is equivalent to $x^2 = x(y + z + w) = xy + xz + xw$. This is a consequence of the distributive law for multiplication over addition, in our terms.

F) (5) Book II of Euclid's *Elements* has often been described as "geometric algebra." What historian that we discussed disputes this and why? Explain *briefly*.

Answer: The most vocal proponent of this point of view is Sabetai Unguru. He claims that Book II should be interpreted as pure geometry and that ascribing algebraic notions to Euclid is a case of *historical conceptual anachronism*, or "Whig history." His point of view is that the mathematics of the past should be understood in its own terms, not in our terms.

G) (5) What major innovation do we see in Diophantos' *Arithmetica*? Explain briefly.

Answer: The major innovation in Diophantos is definitely his rudimentary symbolic notation for algebraic equations and the discussion of solution methods in terms of operations on that symbolic form. He used symbols for the unit (the number 1), the unknown quantity, for its square, and other powers, together with symbols for subtraction to write equations to be manipulated and solved, much as we do today.

H) (5) Where does our word *algebra* come from historically? Explain briefly.

Answer: The word "algebra" is a corruption of an Arabic word from the name of one of Al-Khwarizmi's books, the *Hisab al-jabr w'al-muqabla*. The meaning of the Arabic *al-jabr* in English is something like "restoration."

I) (5) Approximately when and where was Al-Khwarizmi active?

Answer: He was active around 800 CE in the city of Baghdad in present-day Iraq. This was the capital of the Abassid caliphate, and the site of a major scholarly effort where mathematical texts from Greece and India were being translated into Arabic, studied, and then extended.

I) (5) Why did Al-Khwarizmi need to handle "squares equal roots and numbers" quadratic Equations separately from "squares and roots equal numbers" quadratic equations? Explain briefly.

Answer: The question is referring to quadratic equations $ax^2 = bx + c$ (squares equal roots and numbers) versus equations of the form $ax^2 + bx = c$ (squares and roots equal numbers). The reason Al-Khwarizmi (and many later mathematicians as well) handled these separately was that our notion of *negative numbers* had not been invented yet, so it was not the case that both equations could be put in the same form by subtracting all terms to one side and writing $ax^2 + bx - c = 0$ or $ax^2 - bx - c = 0$.

J) (5) What roles did medieval European mathematicians such as Gerbert of Aurillac, Gerard of Cremona, Robert of Chester, and Leonardo of Pisa play in the development of algebra?

Answer: All of these mathematicians were involved in one way or another in transmitting the developments of algebra made by the Islamic mathematicians back to Western Europe, either by teaching (Gerbert), translating Arabic works into Latin (Gerard and Robert), or by writing influential textbooks that popularized algebra and the Hindu-Arabic numerals (Leonardo, also known as "Fibonacci").

K) (5) In what ways were the techniques presented by Viète in his *Introduction to the Analytical Art* an advance over what Diophantos had done with some of the same problems? Explain briefly.

Answer: Viète made systematic use of letters to represent unknowns, and known but arbitrary numbers in his equations. By doing this, he was able to show that Diophantos' solutions, even though they were presented by way of specific examples, were really general solutions of the problems in question (as we had done as well using modern algebra). In other cases, such as the solution of Diophantos' Proposition 8 from Book II, he clarified what was going on by relating the problem to a geometric question about sides of right triangles (Pythagorean triples).

L) (5) The approach to doing geometry introduced by Descartes in *La Géometrie* is often called "analytic geometry" today. What is the historical explanation for this? To what ancient Greek mathematician's work is the word "analytic" primarily referring?

Answer: Historically, this came because mathematicians at this time, including Viète and Descartes were very heavily influenced by Book VII of the *Mathematical Collection* of Pappus of Alexandria, which had been translated from Greek into Latin for the first time in the mid-16th century. Pappus' ideas about *analysis* as a way of discovering mathematical results (see the quotation in question II, part F below) were seen as largely parallel to what was done in posing an algebraic equation and solving for the unknowns.

II. Identifications. For any 4 of the following 6 graphics, texts, or formulas, give the name of the period or the mathematician who would be most closely identified with that item, and explain the meaning briefly. If you submit solutions for more than 4, only the best 4 will be used in computing your score on this question.

A) (5)

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Answer: This is Nicolo Fontana's ("Tartaglia's") solution of the cubic equation $x^3 + px = q$. It was published by Girolamo Cardano in his *Ars Magna* (attributed properly to Tartaglia). But for whatever reason (maybe it's that mathematicians are generally terrible historians!) this is also a special case of what are also called "Cardano's equations."

B) (5) $\Delta^{\Upsilon} 3 \varsigma 4 \Lambda M^o 4$

Answer: This is a sample of a (slightly reworked) version of one of Diophantos' symbolic expressions, namely $3x^2 + 4x - 4$. The ς is the symbol for the unknown, the Δ^{Υ} is the square of the unknown, the M^o is the symbol for the unit, and the Λ is the symbol for

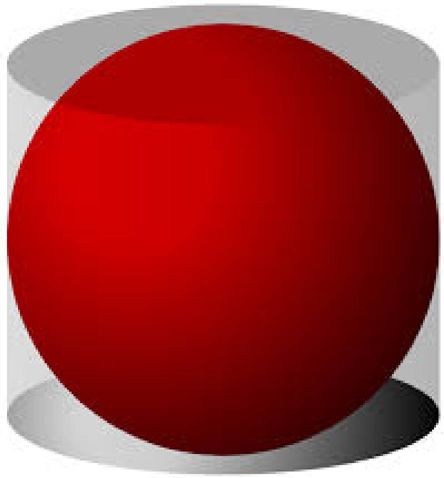

Figure 1: Figure for Question II, part C

subtraction. If this were in its original form, the numbers $3, 4$ would be expressed in the alphabetic Greek format for numbers, though, not the Hindu-Arabic numerals.

C) (5) See Figure 1 above.

Answer: This is the design that Archimedes had inscribed on his tombstone. It shows a sphere inscribed in a cylinder with radius equal to the radius of the sphere and height equal to twice the radius of the sphere. Archimedes had shown that the volume of the sphere was $2/3$ times the volume of the cylinder. He was so proud of this result that he wanted to be associated with it even after he died.

D) (5) "The supreme and everlasting law of equations or proportions, which is called the law of homogeneity because it is conceived with respect to homogeneous magnitudes, is this: Only homogeneous magnitudes are to be compared with one another."

Answer: This is the statement of the Principle of Homogeneity from Viète's *Introduction to the Analytical Art*.

E) (5) "I find nothing here so difficult that it cannot be worked out by anyone at all familiar with ordinary geometry *and algebra*, who will consider carefully all that is set out in this treatise."

Answer: This is from René Descartes' *La Géometrie*, expressing his conviction that his method for studying geometric problems *by means of algebra* is easy enough for anyone to learn and apply to the problems of the 3-line and 4-line loci from Apollonius. This is of course what you did on the final discussion(!)

F) (5) "*Analysis*, then takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is known as the result of synthesis [i.e. things proved before]. For in analysis, we admit that which is sought as if it were already done and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by retracing our steps, we come upon something already known or belonging to the class of first principles ... "

Answer: This is a quotation from Book VII of the *Mathematical Collection* of Pappus of Alexandria, explaining the method of *analysis* for solving mathematical questions.

III. *Essay* (40) Although this has mainly been a course about the historical development of algebra, when you look back at what we have learned, you should see that one consistent theme throughout has been the ways that algebra and geometry have been connected with one another, and the changing ways that mathematicians have thought about that relationship. Discuss the relationship between algebra and geometry in each of the following periods, or mathematical works.

(1) The Old Babylonian period and problem texts such as YBC 6967. Is there a difference of opinion about how to interpret what is algebra and what is geometry there?

(2) The presentation of elementary mathematics in Euclid's *Elements*, especially in Book II.

(3) The *Arithmetica* of Diophantos.

(4) The *Hisab al-jabr w'al-muqabala* of Al-Khwarizmi.

(5) *La Géometrie* of René Descartes.

In teaching algebra today, which way of dealing with algebra and geometry do you think would help students most to learn the subject? And would knowing some of this history help?

Model Response:

(1) There has been a difference of opinion about whether the Old Babylonian problem texts represent algebraic or geometric thinking. Early historians of mathematics such as Otto Neugebauer thought that the Babylonians were thinking algebraically and even claimed that that was the main feature of their mathematics. However, more recent historians such as Jens Høyrup have argued that even though we might be able to interpret what they did as an algebraic procedure, they didn't yet have the conceptual infrastructure in place to be thinking that way. Instead, Høyrup proposes that they were doing "cut and paste" geometry on the rectangle with sides given by the reciprocal pair. Høyrup's interpretation is generally accepted today in place of Neugebauer's.

(2) In Euclid, Book II has often been described as "geometric algebra" because it is possible to express the results of each of the propositions by algebraic equations (as in question I, part E above). However, here too there is some controversy because some historians (including especially Sabetai Unguru) have claimed that Euclid is also thinking purely geometrically in Book II. Moreover, Euclid has in effect presented two forms of these

facts–one for magnitudes in Book II, and later ones for numbers in the "number theory books" in Books VII, VIII, IX. In effect, Euclid has done algebra in geometric form for geometric magnitudes in Book II, but he has treated the parallel algebraic properties of numbers completely separately.

(3) Diophantos takes this separation one step farther and he doesn't really discuss geometry at all in the *Arithmetica*. His work is all about solving algebraic equations for unknown numbers.

(4) Al-Khwarizmi is mainly interested in the algebraic techniques to solve equations, but he gives "justifications" or proofs for what he does by means of geometric diagrams. (In some cases, in fact, what he is doing is applying the geometry from Book II in Euclid to justify the algebra with numbers).

(5) Finally Descartes, working at a time when the algebraic advances of Al-Khwarizmi, the Renaissance Italians, etc. have been throroughly assimilated, sees that *algebra* can serve as a tool in (or even a replacement for) the sort of geometric *analysis* that he has learned from Pappus. Note that if we think the Babylonians were doing something like using geometry to understand algebra, then Descartes has turned the tables completely – he's using algebra to understand geometry(!)

How you come down on the remaining questions is essentially a matter of opinion. But I hope you will try to justify what you say by using your own experience as a mathematics student, together with what we have learned.