College of the Holy Cross, Fall 2016
MONT 106Q – Mathematical Thinking
Solutions for Midterm Exam, November 4, 2016

I. A) (10) Express the base 2 number $(1010110)_2$ in base 10.

Answer: This is the number

$$1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 86$$

in base 10.

B) (10) Express the base 10 number 137 in base 2.

Answer: This is

$$1 \times 128 + 0 \times 64 + 0 \times 32 + 0 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1 = 10001001$$

in base 2.

II. Compute "the Egyptian way"

A) (10) 54×763.

Answer: We double repeatedly starting from 763:

$$\begin{aligned} 1 \times 763 &= 763 \\ *2 \times 763 &= 1526 \\ *4 \times 763 &= 3052 \\ 8 \times 763 &= 6104 \\ *16 \times 763 &= 12208 \\ *32 \times 763 &= 24416 \end{aligned}$$

Since the next power of 2 is 64, which is bigger than 54 we stop here. Then $54 = 32+16+4+2$ (these are the numbers marked with asterisks above). So the product

$$54 \times 763 = 24416 + 12208 + 3052 + 1526 = 41202.$$

B) (10) $1332 \div 12$ (that is, "calculate with 12 to yield 1332")

Answer: We repeatedly double starting from 12:

$$\begin{aligned} *1 \times 12 &= 12 \\ *2 \times 12 &= 24 \\ *4 \times 12 &= 48 \\ *8 \times 12 &= 96 \\ 16 \times 12 &= 192 \\ *32 \times 12 &= 384 \\ *64 \times 12 &= 768 \end{aligned}$$

We stop there since the next doubling would take us past 1332. Then

$$1332 = 764 + 384 + 96 + 48 + 24 + 12$$

So the quotient comes from the marked numbers in the table

$$1332 \div 12 = 64 + 32 + 8 + 4 + 2 + 1 = 111$$

III. Short answer. Answer any four of the following. If you answer more than four, the best four will be used.

A) (5) What base 10 number is represented by these Egyptian hieroglyphs?

   Answer: The number is 347

B) (5) What base did the Maya use in their number system? How would a Maya scribe have represented the number 151?

   Answer: They used a "mixed" base-20 system (meaning that numbers were expressed as sums of multiples of $1, 20, 20 \times 18, 20^2 \times 18, \ldots$, rather than purely in powers of 20). The number $151 = 7 \times 20 + 11$, so the Mayan form is:

C) (5) What are the principal surviving records of Egyptian mathematics and what are their approximate dates?

   Answer: There are two surviving papyri of mathematical problems and their solutions known as the Moscow and Rhind mathematical papyri. The Moscow papyrus is thought to come from about 1800BCE. The Rhind Papyrus is slightly later, about 1650BCE, although it is thought to be copied from earlier work.

D) (5) What was special about the Egyptian way of dealing with fractions?

   Answer: They dealt almost entirely with unit fractions – that is, fractions of the form $\frac{1}{n}$. They employed tables of sums of unit fractions equal to other fractions like $\frac{2}{n}$ to perform arithmetic with more general fractions.

E) (5) What was the *meaning* of the numerical information in the section of the Maya *Dresden Codex* that we studied in Discussion 1?

   Answer: This was a table of number of days between successive lunar eclipses. It apparently served as both a record of a particular sequence of eclipses and as a way to predict patterns of lunar eclipses in the future.

F) (5) The "months" in the Maya *tzol'kin* calendar are, in order,

1.	Imix	11.	Chuen
2.	Ik	12.	Eb
3.	Akbal	13.	Ben
4.	Kan	14.	Ix
5.	Chicchan	15.	Men
6.	Cimi	16.	Cib
7.	Manik	17.	Caban
8.	Lamat	18.	Etznab
9.	Muluc	19.	Ahab
10.	Oc	20.	Ahau

What are the next 3 days after the day 11 Ahab in this system?

Answer: They are 12 Ahau, 13 Imix, 1 Ik.

IV. Essay. (40) Explain in detail two examples of mathematical problems that Christopher Boone discusses in *the curious incident of the dog in the night-time*. What do those problems mean for him? More generally, what is the role of mathematics in his life? Is it fair to say that he relates to the world in mathematical terms? Why or why not?

Model Response: (Note: any two examples are OK, these are just the two that stood out the most for me.) Two examples of mathematical problems discussed by Christopher Boone in *the curious incident of the dog in the night-time* are the "Monty Hall" problem and the problem about Pythagorean triples from the A-level examination he discusses in the Appendix. The Monty Hall problem deals with a situation from the TV show "Let's Make a Deal." The host, Monty Hall, would play a game with contestants on the show where three doors concealed a car and two worthless prizes like goats. The contestant would select one of three doors, then Monty Hall would reveal one of the goats and ask the contestant if they wanted to change their original choice. The question was whether it was better strategy to always change or to always stay with the original choice. This question and a proposed solution that it was better to change appeared in the column by Marilyn Vos Savant in the 1980's. A number of people, including many professional mathematicians disagreed. But Christopher is able to reason correctly that switching doors is the better strategy. He explains that he likes this problem because it shows that mathematics is *not* just "cut and dried" and "safe" as his teacher Mr. Jeavons had said. There are genuinely surprising and difficult things in the subject. But his (Christopher's) insight and skill allow him to reason through to correct answers, and he feels powerful for that reason.

The second problem asks one to show that any triangle with sides of the form $(n^2 - 1, 2n, n^2+1)$ for some positive integer $n \geq 2$ must be a right triangle. Christopher solves this very carefully and completely by showing that the Pythagorean theorem equation follows:

$$(n^2 + 1)^2 = n^4 + 2n^2 + 1 = (n^2 - 1)^2 + (2n)^2$$

Since the relation $c^2 = a^2 + b^2$ implies the triangle has a right angle opposite the side of length c, this shows the triangle has a right angle. This problem shows how well Christopher understands basic secondary-school mathematics and that he really is ready to go on to study more advanced subjects. His biggest success in this novel is taking and passing this examination. It again shows his comfort and skill in dealing with mathematical ideas.

In both of these examples and in general, we can see that Christopher is most at home thinking in mathematical terms–that is where his brain works best, where he is at his most comfortable, and where he is most himself. His first inclination is often to try to figure out situations where he finds himself by thinking in mathematical or logical terms. So yes, it is pretty fair to say he relates to the world mostly in mathematical terms. Part of what makes his life difficult is that he cannot do this all of the time–situations like the impasse between his parents, or his inability to handle enviroments with too many stimuli don't allow him to use his strengths in the same way.