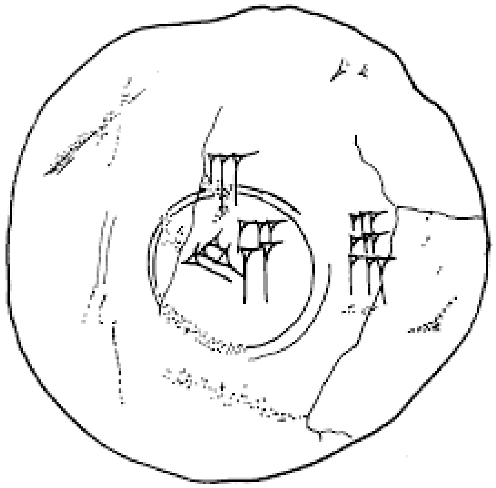

Figure 1: The Old Babylonian tablet YBC 7302

College of the Holy Cross, Fall 2019
MATH 110-02 – Algebra Through History
Solutions for Midterm Exam, October 25, 2019

I. A) (15) Refer to Figure 1 above showing an Old Babylonian practice tablet from a scribal school in the city of Susa. It shows three separate numbers in the cuneiform base-60 script. What are those three numbers?

Solution: The numbers are 3 (over the circle), 45 (inside the circle), and 9 (outside the circle to the right).

B) (5) From the context (the rough circle with the numbers), it is surmised that this shows a calculation of an approximate area of a circle. Interpret the number written over the circle as the circumference, and the number inside the circle as the area. *Question:* What value is being used for the constant π? Hints: (1) The number inside is not a whole number; you want to think of it as the first digit after the ; in the fractional part. Useful information: the area of a circle of radius r is $A = \pi r^2$ and the circumference is $C = 2\pi r$, so the area is $A = \frac{C^2}{4\pi}$ in terms of the circumference.

Solution: With the hint about the number inside the circle, this represents

$$\frac{45}{60} = \frac{3}{4}$$

Assuming that the area was computed using the formula in terms of the circumference,

$$\frac{3}{4} = \frac{9}{4\pi} \Rightarrow \pi = 3.$$

(This was a common approximation for π before more accurate values such as $\pi \doteq 3.1416$ were developed.)

II.

A) (5) In Greek mathematics, what did it mean to say that that two magnitudes are *incommensurable*?

Solution: If the quantities are x, y, saying they are incommensurable means that there are no positive integers m, n such that $n \cdot x = m \cdot y$, or equivalently that the ratio $\frac{x}{y}$ is not a rational number $\frac{m}{n}$.

B) (15) Give the proof the ancient Greeks found for the fact that the diagonal of a square of side 1 is not commensurable with the side.

Solution: If the side of the square is 1, then the diagonal is $\sqrt{2}$ by the Pythagorean theorem. If $\sqrt{2} = \frac{m}{n}$ for some integers m, n, then we can take the fraction in *lowest terms*; that is, we can assume that m, n have no common factors. Squaring, we get $m^2 = 2n^2$. This shows m^2 is even and it follows that m is even too, since the square of an odd number is odd. Hence $m = 2k$ for some integer k. Substituting, we get $4k^2 = 2n^2$. So then $n^2 = 2k^2$, and as before n^2 must be even, so n is also even. But this is a contradiction since we assumed m, n had no common factors, and we have shown that they must both be divisible by 2. Hence $\sqrt{2}$ and 1 are incommensurable.

III. Short answer. Answer any four of the following. If you answer more than four, you can earn some Extra Credit points.

A) (5) One of Diophantos' propositions contains the following expression $\Delta^{\Upsilon}4M^{o}16\Lambda\varsigma16$ (containing one unknown). Rewrite this in modern notation, calling the unknown x.

Solution: This is Diophantos' way of writing

$$4x^2 + 16 - 16x \text{ or } 4x^2 - 16x + 16.$$

The $\Delta^{\Upsilon}4$ is the x^2; the $M^{o}16$ is the 16 and the $\Lambda\varsigma16$ is the $-16x$.

B) (5) What are the approximate dates of the YBC 6967 and YBC 7289 tablets? About when did Euclid live?

Solution: The Babylonian tablets are from somewhere between 2000BCE and 1600BCE, probably toward the middle of that range. Euclid lived some time around 300BCE.

C) (5) What is "Fermat's Last Theorem" and what is its connection with Diophantos? When and by whom was this finally solved?

Solution: "Fermat's Last Theorem" is the statement that the equation $x^n + y^n = z^n$ has no integer solutions with x, y, z all nonzero when $n \geq 3$. The connection with Diophantos is that Pierre Fermat wrote this in his copy of a Latin translation of Diophantos' *Arithmetica*. He said he had found a "marvelous proof" but that the margin of the page was too small to contain it. It is now thought that he probably did not have a complete proof. This was finally proved by Andrew Wiles (and Richard Taylor) and announced to the world in 1993.

D) (5) In what way is the solution of the problem given on the YBC 6967 tablet related to our modern quadratic formula?

Solution: Probably the best way to say this is that the step by step solution given on YBC 6967 produces one of the roots of the quadratic equation $x^2 = 60 + 7x$, or $x^2 - 7x - 60 = 0$ by a method that is equivalent to (a slight rearrangement of) the quadratic formula we use:

$$x = \frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 + 60}.$$

But the Babylonians were probably thinking of this as a series of steps in a "cut-and-paste" geometry construction as explained by Jens Høyrup.

E) (5) Proposition 4 in Book II of Euclid's *Elements* says: *If a straight line is cut at random then the square on the whole line is equal (in area) to the sum of the squares on the pieces together with twice the rectangle contained by the two pieces.* What modern algebraic equation is equivalent to this? Call the lengths of the two pieces x, y.

Solution: This says

$$(x + y)^2 = x^2 + y^2 + 2xy.$$

F) (5) What would be the hardest arithmetic operations to carry out in the Old Babylonian base-60 number system? How did they keep track of all the information they needed for these?

Solution: The hardest arithmetic operations would be multiplication and division. The Babylonians dealt with the large number of possible products of pairs of base-60 digits by means of large tables of numerical data. Those would have been used by the scribes for these calculations.

IV. (40) Essay. You have the choice of responding to *either prompt 1 or 2*. State which one you have chosen at the start of your essay.

1) A certain older history of mathematics says, flatly, that "the distinguishing feature of Babylonian mathematics is its algebraic character." Of the historians we have mentioned, who would agree with this claim and who would disagree? Explain using the interpretations your historians would give for the YBC 6967 problem of (what we would phrase as) solving the equation $x = 60/x + 7$.

Response: (Note: The following level of mathematical detail is sufficient; if you said more, that is fine too!) Of the historians we discussed, Otto Neugebauer would be the one who would agree most strongly with this statement. He saw the YBC 6967 solution as an application of the algebraic identity

$$\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 = xy$$

(taking x and $60/x$ to the be "reciprocal pair", so $x-y=7$). Neugebauer was one of the first mathematicians to study Old Babylonian mathematical tablets and his point of view was very influential up until the 1970's or so. At that point, Jens Høyrup proposed that the steps in the solution of the YBC 6967 would be more properly interpreted as a series of "cut and paste" geometric operations on a rectangle with sides x and $60/x$ rather than as algebra in the modern sense. He and Eleanor Robson have pointed out that the language used in the solution seems to be pointing toward the geometric interpretation. Moreover, the context of these problem texts in the Old Babylonian scribal schools seems to be closer to geometric questions about areas than it does to what we would call algebra. The subtle point is that *both interpretations are correct in the mathematical sense*. Historians such as Sabetai Unguru would say, though, that Neugebauer's view is wrong historically. It ascribes a modern understanding of algebra to the ancient Babylonians, when that is probably not appropriate. In other words, that would be a case of *conceptual anachronism* or "Whig history."

2) George G. Joseph, the author of another book on the non-European roots of modern mathematics called *The Crest of the Peacock*, offers this overall evaluation of the ultimate impact of Greek geometry: "There is no denying that the Greek approach to mathematics produced remarkable results, but it also *hampered* the subsequent development of the subject. ... Great minds such as Pythagoras, Euclid, and Apollonius spent much of their time creating what were essentially abstract idealized constructs; how they arrived at a conclusion was in some way more important than any practical significance." First, what does the last sentence mean? Would this criticism seem to be apt for Diophantos' *Arithmetica* as well? Is it necessary for all the mathematics we learn and do to have practical usefulness or significance?

   One Response: The last sentence in the quote from Joseph's book is referring to the fact that in texts like Euclid's *Elements*, much attention is paid to the process of proving general statements about the abstract geometric properties of triangles, parallelograms, squares, etc. from the postulates and previously-proved propositions. As we have seen Euclid devotes no attention whatsoever to possible uses of mathematics (geometry or algebra) to solve real-world problems. He seems in particular to care more about how statements are proved ("how they arrived at a conclusion") than in potential applications or practical consequences.

   Moreover, Joseph is saying that this focus on proof and on the logical structure of mathematical deduction *held Greek and later mathematics back* because it divorced mathematics from the applications that would have lead to other new ideas and discoveries. So that aspect of Greek mathematics was not, ultimately, healthy for the subject, according to Joseph. For Joseph, and for many modern mathematicians, a lot of really interesting mathematics comes from practical problems and the questions raised by the sciences and other areas of inquiry. So as important as Euclid was for setting standards of logical "rigor," his side of the subject is not the only one (or maybe even the most important one).

   The same criticism might apply at least partly to Diophantos' *Arithmetica*. Even though the subject of his work is solving equations in integers or rational numbers and not geometrical relationships, Diophantos does not indicate any practical use for the solution techniques he discusses either. It's probably easier, though, to see the connection between what he does and techniques for solving equations that would come from practical situations. And the introduction of the symbolic way of representing algebraic expressions certainly influenced later developments.

[*My personal "take" on the last question:*] Whether or not all mathematics should have practical applications is a subtle question for several reasons. First, new mathematical ideas are often found before their eventual applications and the applications often come later, when people have really understood what the mathematics is saying. If people were only studying things that had clear practical applications in the present, then the subject might also be hampered (as Joseph says), but in a different way. Namely, the ideas to solve a practical problem might not have been developed yet when they are really needed. Second, some mathematicians are attracted to the "pure" side of the subject exclusively; they don't actually care whether there are applications or not. But on the other hand, from the ethical point of view, it's hard to justify a life spent doing something that has no connection with the real world, no relevance for the issues confronting people in the present, and that is essentially worthless(!) And as I said above, most current mathematicians would say that the subject is invigorated by the real-world applications that people are developing. As is true for many things, it is probably better not to go too far in either direction (either only studying pure mathematics or only studying applied mathematics). The ideal mix is probably "some of each!"