Mathematics 243, section 3 – Algebraic Structures
Solutions for Problem Set 4
due: October 5, 2012

'A' Section

1. Consider the following relations defined on the set $\mathbb{Z}$. In each case, say whether the relation is reflexive, symmetric, transitive. Justify your answers.

   a. xRy if and only if $(-1)^x = (-1)^y$

   Solution: We see
$$(-1)^x = \begin{cases} +1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases}$$
   Hence xRx is true for all x, so R is reflexive. Similarly, if xRy then x, y are both even or both odd. So yRx follows, and R is symmetric. Finally R is transitive since xRy and yRz imply x, y, z are either all even or all odd.

   b. xRy if and only if $x \cdot y \geq 0$

   Solution: xRx is true for all $x \in \mathbb{Z}$, since $x^2 \geq 0$. So R is reflexive. Similarly, R is symmetric since multiplication in $\mathbb{Z}$ is commutative, so $xy = yx$ and if $xy \geq 0$, $yx \geq 0$ too. R is *not transitive* since for instance $(1)R(0)$ is true and $(0)R(-1)$ is true, but $(1)R(-1)$ is not true.

   c. xRy if and only if $|x - y| \leq 2$

   Solution: This is similar to b in the sense that this relation is reflexive since $|x-x| = 0 \leq 2$ for all x, and symmetric since $|x - y| = |y - x|$ for all $x, y \in \mathbb{Z}$. But it is not transitive, since for example $(-2)R(0)$ is true since $|(-2) - 0| = 2 \leq 2$ and $(0)R(2)$ is true since $|0 - 2| = 2 \leq 2$, but $(-2)R(2)$ is not true since $|-2 - 2| = 4$.

   d. xRy if and only if x has the same number of base 10 digits as y, ignoring signs of x, y.

   Solution: This is reflexive – every x has the same number of base 10 digits as itself. Similarly, it is symmetric since if x has the same number of digits as y, then y has the same number of digits as x. It is also transitive, since if x and y have the same number of digits and y and z have the same number of digits, then so do x and z.

   e. xRy if and only if the sum of the base 10 digits of x is the same as the sum of the base 10 digits of y, ignoring signs of x, y.

   Solution: This is reflexive, symmetric, and transitive as in part d (just replace "number of digits" by "sum of digits" everywhere).

2. Which of the relations in question 1 are equivalence relations? For those that are, say exactly which integers make up the equivalence class [11] using correct set notation.

Solution: From the answers to question 1, the relation in a is an equivalence relation and

$$[11] = \{m \in \mathbb{Z} \mid m \text{ is odd}\}$$

The relation in d is also an equivalence relation. The equivalence class of [11] consists of all positive or negative integers that have the exactly two digits in their base 10 forms:

$$[11] = \{\pm 10, \pm 11, \pm 12, \ldots, \pm 99\}.$$

Finally, the relation in part e is also an equivalence relation. The class of 11 consists of all numbers whose digits add up to 2:

$$[11] = \{\pm 2, \pm 20, \pm 200, \ldots, \pm 11, \pm 101, \pm 110, \ldots\}$$

(Note: there can be any number of 0 digits in these.)

3. Let R be the relation on $\mathbb{Z}$ defined by xRy if and only if $4x - 15y$ is a multiple of 11. Show that R is an equivalence relation and describe all of the equivalence classes for R.

   Solution: For all integers x, xRx is true since $4x - 15x = 11 \cdot (-x)$ is a multiple of 11. Hence R is reflexive. If xRy, then $4x - 15y = 11k$ for some integer k. But then $4y - 15x = -(4x - 15y) - 11x - 11y = 11(-k - x - y)$ is a multiple of 11, so yRx follows. Hence R is symmetric. Finally, if xRy and yRz then $4x - 15y = 11k$ and $4y - 15z = 11\ell$ for some integers k, ℓ. But then $4x - 15y + 4y - 15z = 11(k + \ell)$, so $4x - 15z = 11(k + \ell + y)$ is also a multiple of 11. It follows that xRz is true, so R is transitive. This shows that R is an equivalence relation.

   The equivalence class of any $x \in \mathbb{Z}$ is the set

$$[x] = \{y \mid xRy\} = \{y \mid 4x - 15y = 11k, \text{ for some integer } k\}$$

   It can be seen that there are only 11 different classes:

$$\begin{aligned}
[0] &= \{\ldots, -22, -11, 0, 11, 22, \ldots\} \\
[1] &= \{\ldots, -21, -10, 1, 12, 23, \ldots\} \\
&\vdots \\
[10] &= \{\ldots, -12, -1, 10, 21, 32, \ldots\}
\end{aligned}$$

4. Decide whether each of the following statements is true. For those that are true, give a short proof using the postulates for $\mathbb{Z}$ given in §2.1 of the text. For those that are false, give a counterexample.

   a. If $xy = xz$ for integers x, y, z, then $y = z$.

      Solution: This is false: let $x = 0$, $y = 1$, $z = 2$.

   b. If $x < y$, then $x^2 < y^2$.

Solution: This is also false: Let $x = -3$ and $y = 2$. Then $x < y$, but $x^2 > y^2$.

c. If $z - x < z - y$, then $y < x$.

Solution: This is true. If $z - y > z - x$, then $(z - y) - (z - x) \in \mathbb{Z}^+$. But that says $x - y \in \mathbb{Z}^+$, so by the definition of the order relation $x > y$.

'B' Section

1. In class we showed that the distinct equivalence classes of an equivalence relation R on a set A give a partition of A. Conversely, suppose

$$A = \bigcup_{\lambda \in \mathcal{L}} A_\lambda$$

is a partition of A. Show that the relation R on A defined by aRa' if and only if a, a' are both elements of the same subset A_λ is an equivalence relation.

Solution: Let $a \in A$. Then a is contained in only one of the A_λ, since they form partition. That implies aRa is true, and R is reflexive. Next, suppose xRy. That means that x and y are in the same set in the partition, so it follows that y and x are also in the same set. Hence yRx is also true and R is symmetric. Finally, if xRy and yRz are both true then x, y, z are all in the same set in the partition so xRz is also true. This shows R is transitive too, hence an equivalence relation.

2. In both parts of this problem, you will be working in $\mathbb{Z}$, using the postulates from §2.1

a. Show that if $x \cdot y = 0$, then $x = 0$ or $y = 0$. (Hint: Argue by contraposition. By the trichotomy postulate 4, if $x \neq 0$, then $x \in \mathbb{Z}^+$ or $-x \in \mathbb{Z}^+$, and the same is true for y.)

Solution: We want to show that if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$. By postulate 4 in the book's numbering, if $x \neq 0$, then $x \in \mathbb{Z}^+$ or $-x \in \mathbb{Z}^+$. Similarly, $y \in \mathbb{Z}^+$ or $-y \in \mathbb{Z}^+$. There are four possible combinations of statements that can be true here. If $x \in \mathbb{Z}^+$ and $y \in \mathbb{Z}^+$, then $xy \in \mathbb{Z}^+$ by postulate 4b. If $-x \in \mathbb{Z}^+$ and $y \in \mathbb{Z}^+$, then $(-x)y \in \mathbb{Z}^+$ by postulate 4b. But $(-x)y = -(xy) \in \mathbb{Z}^+$ by Theorem 2.2, and then $xy \neq 0$. Similarly, $x \in \mathbb{Z}^+$ and $-y \in \mathbb{Z}^+$, then $x(-y) \in \mathbb{Z}^+$ by postulate 4b. But $x(-y) = (-y)x = -(yx) \in \mathbb{Z}^+$ by Theorem 2.2, and then $yx = xy \neq 0$. Finally, $-x \in \mathbb{Z}^+$ and $-y \in \mathbb{Z}^+$, then $(-x)(-y) \in \mathbb{Z}^+$ by postulate 4b. But $(-x)(-y) = xy \in \mathbb{Z}^+$ by Exercise 5 in this section of the book (proof follows same idea as the proof $(-1)(-1) = 1$ done in class) and then $xy \neq 0$.

b. From part a, deduce the cancellation law in $\mathbb{Z}$: If $x \cdot y = x \cdot z$ and $x \neq 0$, then $y = z$.

Solution: If $x \cdot y = x \cdot z$, then $x \cdot (y - z) = 0$. We are assuming $x \neq 0$, so part a implies $y - z = 0$, and hence $y = z$.

3. Prove by mathematical induction:

a. For all $n \in \mathbb{Z}^+$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: The base case here is $n = 1$. The formula is true in that case because $1^2 = 1 = \frac{1\cdot2\cdot3}{6}$. Now assume the formula is true for $k \in \mathbb{Z}$, and consider the case $n = k+1$. We have, using the induction hypothesis and some algebra:

$$\begin{aligned}
1^2 + 2^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)(2k^2 + k + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6},
\end{aligned}$$

which is what we wanted to show since

$$\frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

b. For all $n \in \mathbb{Z}^+$,

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Solution: The base case here is $n = 1$. The formula is true in that case because $\frac{1}{1\cdot2} = \frac{1}{2} = \frac{1}{1+1}$. Now assume the formula is true for $k \in \mathbb{Z}$, and consider the case $n = k+1$. We have, using the induction hypothesis and some algebra:

$$\begin{aligned}
\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{(k+1)(k+2)} &= \left(\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{k(k+1)}\right) + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2},
\end{aligned}$$

which is what we wanted to show since

$$\frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}.$$

c. For all $n \geq 2$, $n^3 > 1 + 2n$.

Solution: The base case is $n = 2$, and the inequality is true in that case since $8 > 5$. Now assume $k^3 > 1+2k$ and consider $(k+1)^3 = k^3+3k^2+3k+1$ By the induction hypothesis, we see $(k+1)^3 > 1 + 2k + 3k^2 + 3k + 1 > 1 + 2(k+1)$, since $3k^2 + 3k + 1 > 3k + 1 > 2$ for all $k \geq 2$.

d. If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. (Hint: For the induction step, let $A = \{a_1, \ldots, a_k, a_{k+1}\}$. Every subset of A is of one of two types – the ones containing a_{k+1} and the ones not containing a_{k+1}. Count the number of subsets of each type by using the induction hypothesis.)

Solution: When $n = 0$, $A = \emptyset$ has exactly 1 subset, namely $\emptyset$. Therefore $|\mathcal{P}(A)| = 1 = 2^0$. So the base case is established. Now assume that $|\mathcal{P}(A)| = 2^k$ whenever $|A| = k$ and consider $A = \{a_1, \ldots, a_k, a_{k+1}\}$. Following the hint, note that

$$\mathcal{P}(A) = S_1 \cup S_2$$

where $S_1 = \{T \subseteq A \mid a_{k+1} \in T\}$ and $S_2 = \{T \subseteq A \mid a_{k+1} \notin T\}$. By their definitions, $S_1 \cap S_2 = \emptyset$, so

$$|\mathcal{P}(A)| = |S_1| + |S_2|.$$

The subsets in S_2 are in one-to-one correspondence with the subsets of $\{a_1, \ldots, a_k\}$. Hence $|S_2| = 2^k$ by the induction hypothesis. Every subset T in S_1 contains a_{k+1}, so it can be written as $T = T' \cup \{a_{k+1}\}$, where T' is a subset of $\{a_1, \ldots, a_k\}$. Distinct T''s give distinct T's so there are exactly as many elements of S_1 as subsets of $\{a_1, \ldots, a_k\}$. By the induction hypothesis again, that number is 2^k. Hence

$$|\mathcal{P}(A)| = |S_1| + |S_2| = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Therefore $|\mathcal{P}(A)| = 2^n$ for all $n \geq 1$ by induction.

4. The binomial coefficients are the numbers

$$\binom{n}{\ell} = \frac{n!}{\ell!(n-\ell)!}$$

(where $0! = 1$ by convention).

a. Show using the definition that for all ℓ with $1 \leq \ell \leq n$,

$$\binom{n}{\ell} + \binom{n}{\ell-1} = \binom{n+1}{\ell}$$

Solution: We have

$$\binom{n}{\ell} + \binom{n}{\ell-1} = \frac{n!}{\ell!(n-\ell)!} + \frac{n!}{(\ell-1)!(n-\ell+1)!}$$

The common denominator for these two fractions is $\ell!(n-\ell+1)!$, so adding we have

$$\frac{n!}{\ell!(n-\ell)!} + \frac{n!}{(\ell-1)!(n-\ell+1)!} = \frac{n!(n-\ell+1+\ell)}{\ell!(n+1-\ell)!} = \frac{(n+1)!}{\ell!(n+1-\ell)!} = \binom{n+1}{\ell}.$$

b. (The Binomial Theorem) Show by induction that for all $n \geq 1$.

$$(a+b)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} a^\ell b^{n-\ell}$$

(that is, the numbers $\binom{n}{k}$ are exactly the coefficients of the various terms $a^k b^{n-k}$ appearing in the expansion of $(a+b)^n$).

Solution: The base case $n = 1$ follows since $\binom{1}{0} = 1 = \binom{1}{1}$, and $(a+b)^1 = a + b = \binom{1}{0} b + \binom{1}{1} a$.

Now assume the theorem has been proved for $n = k$ and consider the case $n = k+1$. We have by the induction hypothesis,

$$\begin{aligned}(a+b)^{k+1} &= (a+b)^k(a+b) \\ &= \left(\sum_{\ell=0}^{k} \binom{k}{\ell} a^\ell b^{k-\ell}\right)(a+b).\end{aligned}$$

Expand the product on the last line using the distributive law and collect like terms. The coefficients of a^{k+1} and b^{k+1} are both equal to 1 by inspection. These match the formula to be proved since $\binom{k+1}{k+1} = \binom{k+1}{0} = 1$. Now assume $1 \leq \ell \leq k$. The $a^\ell b^{k+1-\ell}$ term comes from

$$b \cdot \left(a^\ell b^{k-\ell} \text{ term in first factor}\right) + a \cdot \left(a^{\ell-1} b^{k-(\ell-1)} \text{ term in first factor}\right).$$

From above (by the induction hypothesis), the coefficient of this term is

$$\binom{k}{\ell} + \binom{k}{\ell-1} = \binom{k+1}{\ell},$$

where the right side comes by applying part a of this question. Therefore

$$(a+b)^{k+1} = \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} a^\ell b^{k+1-\ell},$$

and the result follows by induction.