Mathematics 243, section 3 – Algebraic Structures
Solutions for Exam 1 – September 28, 2012

I. Let A, B, C be any subsets of a universal set U.

A) Show that if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.
Solution: Let $x \in A \cap C$. Then $x \in A$ and $x \in C$. It follows from $A \subseteq B$ and $x \in A$ that $x \in B$ as well. Hence $x \in B$ and $x \in C$, so $x \in B \cap C$. This shows $A \cap C \subseteq B \cap C$.

B) Is it true that $A - B = A - C$ implies $B = C$? If you think the statement is true, illustrate with a Venn diagram. If you think the statement is false, give a counterexample.
Solution: This statement is false because $A - B = A \cap B'$ and $A - C = A \cap C'$. To make a counterexample, we just need to think of sets B and C whose complements intersect A in the same set, but which have different elements. Here's a simple example. In $U = \{1, 2, 3\}$, let $A = \{1\}$, $B = \{2, 3\}$ and $C = \{2\}$. Then $B' = \{1\} = A \cap B'$ and $C' = \{1, 3\}$, so $A \cap C' = \{1\}$. But clearly $B \neq C$.

II. Consider the following statement, where A is a nonempty set. "If $f : A \to B$ is an onto mapping, then for all $b \in B$, the set $f^{-1}(\{b\})$ is nonempty." Give the contrapositive form of this statement. Are the given statement and its contrapositive true or false? Explain.

Solution: The contrapositive form is: "If for some $b \in B$ the set $f^{-1}(\{b\})$ is empty, then f is not an onto mapping." This statement and the original form are equivalent and both true. By definition, f is onto if for all $b \in B$, there exist $a \in A$ such that $f(a) = b$. Those a are precisely the elements of $f^{-1}(\{a\})$, so that set must be nonempty. That is the given form of the statement.

III. Let $f, g : \mathbf{Z} \to \mathbf{Z}$ be the mappings given by

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases} \qquad g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$$

A) Let $S = \{1, 2, 3\}$ and $T = \{0, 3, 6\}$. What is $g^{-1}(T) \cap f(S)$?
Solution: We have $f(S) = \{f(1), f(2), f(3)\} = \{2, 3, 4\}$. Then $g^{-1}(T) = \{x \in \mathbf{Z} \mid g(x) \in T\} = \{0, -1, 6, 5\}$. (Note that $g(x)$ is even for all x, so there are no x that map to $3 \in T$.) So $g^{-1}(T) \cap f(S) = \emptyset$.

B) What is the mapping $f \circ g$?
Solution: By the observation about g, given in the previous part, in computing $(f \circ g)(x)$, we never use the part of the definition of f for odd x, and:

$$(f \circ g)(x) = \begin{cases} 2x - 1 & \text{if } x \text{ is even} \\ 2(x + 1) - 1 = 2x + 1 & \text{if } x \text{ is odd} \end{cases}$$

IV. Consider the binary operation on $\mathbf{Z}$ defined by $x * y = x + xy + y$.

A) Is $*$ associative?

Solution: Let x, y, z be any three integers. We have

$$\begin{aligned}(x * y) * z &= (x + xy + y) * z \\ &= x + xy + y + (x + xy + y)z + z \\ &= x + y + z + xy + xz + yz + xyz\end{aligned}$$

On the other hand, we have

$$\begin{aligned}x * (y * z) &= x * (y + yz + z) \\ &= x + x(y + yz + z) + y + yz + z \\ &= x + y + z + xy + xz + yz + xyz\end{aligned}$$

These are always the same, so $*$ is associative.

B) There is an identity element for $*$. Find it and justify your claim.
Solution: By inspection $e = 0$ acts as an identity for $*$.

V. (15) Let $A \in M_{2\times 2}(\mathbf{Z})$ be the matrix $A = (a_{ij})$ defined by $a_{ij} = i - j$, and let $B \in M_{3\times 2}(\mathbf{Z})$ be the matrix $B = (b_{ij})$ defined by $b_{ij} = i + 2j$. Which of the two products AB, BA are defined? Compute the one(s) that are defined.

Solution: By the given information,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{pmatrix}$$

The sizes match to multiply only for BA and

$$BA = \begin{pmatrix} 5 & -3 \\ 6 & -4 \\ 7 & -5 \end{pmatrix}$$

Extra Credit Which $x \in \mathbf{Z}$ have inverses for the binary operation $*$ from question IV?

Solution: x has an inverse y when $x + xy + y = 0$ (using the identity from part B) of IV. This says $y = \frac{-x}{x+1}$. But this formula gives integer values only when $x = 0$ and $x = -2$. The inverse of 0 is 0 and the inverse of -2 is -2.