Mathematics 243, section 1 – Algebraic Structures
Solutions for Midterm Exam 1 – September 29, 2006

I. For this problem,

$$\begin{aligned} U &= \mathbf{Z} \\ A &= \{x \in \mathbf{Z} : x \text{ is odd and } -10 \le x \le -4\} \\ B &= \{-2, -1, 0, 1\} \\ C &= \{x \in \mathbf{Z} : x < 0\} \end{aligned}$$

and $f : \mathbf{Z} \to \mathbf{Z}$ is the mapping defined by

$$f(x) = \begin{cases} x+3 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases}.$$

A) (10) What is the set $A \cup (B \cap C')$?

Solution: $C' = \{x \in \mathbf{Z} : x \ge 0\}$, so $B \cap C' = \{0, 1\}$, and

$$A \cup (B \cap C') = \{-9, -7, -5, 0, 1\}.$$

B) (10) What is the set $f(B)$?

Solution: $f(B) = \{f(-2), f(-1), f(0), f(1)\} = \{1, -2, 3, 0\}$ (using the first part of the definition to find $f(-2), f(0)$ and the second to get $f(-1), f(1)$).

C) (10) Given: f is a permutation of $\mathbf{Z}$ (you don't need to show this). Find the inverse mapping $f^{-1} : \mathbf{Z} \to \mathbf{Z}$.

Solution: Since $x+3$ is odd when x is even, and $x-1$ is even when x is odd:

$$f^{-1}(x) = \begin{cases} x-3 & \text{when } x \text{ is odd} \\ x+1 & \text{when } x \text{ is even} \end{cases}$$

II. Let $f : A \to B$ be any mapping.

A) (15) Prove that for any subsets T_1 and T_2 of B, $f^{-1}(T_1 \cup T_2) = f^{-1}(T_1) \cup f^{-1}(T_2)$.

Solution: by the definitions of the inverse image and union of sets we have

$$\begin{aligned} x \in f^{-1}(T_1 \cup T_2) &\Leftrightarrow f(x) \in T_1 \cup T_2 \\ &\Leftrightarrow f(x) \in T_1 \text{ or } f(x) \in T_2 \\ &\Leftrightarrow x \in f^{-1}(T_1) \text{ or } x \in f^{-1}(T_2) \\ &\Leftrightarrow x \in f^{-1}(T_1) \cup f^{-1}(T_2) \end{aligned}$$

Hence $f^{-1}(T_1 \cup T_2) = f^{-1}(T_1) \cup f^{-1}(T_2)$ because each of these sets is contained in the other.

B) (10) Is $f^{-1}(f(S))$ always equal to S for all subsets $S \subseteq A$? If you say no, give a counterexample; if you say yes, say why.

Solution: This is *not* always true. It can fail when the mapping f is not injective (one-to-one). For instance, let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Then if $S = \{1\}$, we have $f(S) = \{1\}$. But $f^{-1}(f(S)) = \{-1, 1\}$, since both $f(-1) = 1$ and $f(1) = 1$.

III. Let $*$ be the binary operation on $\mathbf{R}$ defined by $x * y = xy - 2x$.

A) (10) Show with specific counterexamples that $*$ is *not associative*, and *not commutative.*

Solution: $*$ is not associative because for instance $(1 * 2) * 1 = (2 - 2) * 1 = 0 * 1 = 0$, but $1 * (2 * 1) = 1 * (2 - 4) = 1 * (-2) = -2 - 2 = -4$. Similarly, $*$ is not commutative: $1 * 2 = 0$ but $2 * 1 = -2$.

B) (15) An element e is said to be a *left identity* for a binary operation if $e * x = x$ for all x. Similarly, e is a *right identity* if $x * e = x$ for all x. Does the $*$ operation have a left identity? Does it have a right identity? If so, say what e is in each case. If not, explain how you reach your conclusions.

Solution: $*$ *does have a right identity* $e = 3$ since $x * 3 = 3x - 2x = x$ for all x. On the other hand, it does not have a left identity because if $e * y = y$ for all y, then we have $ey - 2e = y$ for all $y \in \mathbf{R}$ This equation says $e = \frac{y}{y-2}$, so while there is some e that works for each y individually, there is no single e that works for all y.

IV. In both parts of this problem,

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}.$$

A) (10) Compute the matrix product AB or say why it doesn't exist.

Solution: A and B are 2×2 matrices, so the product is defined, and

$$AB = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(the 2×2 identity matrix). Note that

$$BA = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

also.

B) (10) Define a mapping

$$F : M_{2\times 2}(\mathbf{R}) \to M_{2\times 2}(\mathbf{R})$$

for X a 2×2 matrix by by $F(X) = BX$ (matrix product). Prove that F is one-to-one (injective) and give its left inverse mapping.

Solution: We show F is injective first. Assume $F(X) = F(Y)$ for two matrices X, Y. Then $BX = BY$. Multiply A on the left of both sides in this equation: $A(BX) = A(BY)$. By associativity of matrix multiplication, this says $(AB)X = (AB)Y$. Since $AB = I_2$ (the 2×2 identity matrix), this implies $I_2X = I_2Y$, so $X = Y$.

The left inverse mapping of F is $G : M_{2\times 2}(\mathbf{R}) \to M_{2\times 2}(\mathbf{R})$ defined by $G(X) = AX$. This follows because for all X

$$(G \circ F)(X) = G(F(X)) = A(BX) = (AB)X = I_2X = X$$

(The fact that F is injective also follows from the fact that there is a left inverse.)