Mathematics 136, section 2 – Advanced Placement Calculus
Solutions for Exam 2 – November 4, 2009

I. Consider the function $f(x) = x^3 + 6x^2 - 15x$.

A) (10) Find all the critical numbers of f and classify each as a local maximum, local minimum, or neither.

Solution: We have $f'(x) = 3x^2 + 12x - 15 = 3(x + 5)(x - 1)$. This is defined for all x and zero only at $x = -5, 1$ so those are the critical numbers. We have $f''(x) = 6x + 12$. Hence $f''(-5) = -18$ and $f''(1) = 18$. By the Second Derivative Test, $x = -5$ is a local maximum, and $x = 1$ is a local minimum. (Note: this can also be determined by the First Derivative Test, finding the intervals where f' is positive ($x < -5$ and $x > 1$) and the interval where f' is negative ($-5 < x < 1$).

B) (5) What are the absolute (global) maximum and minimum values of $f(x)$ on the interval $[0, 3]$?

Solution: The critical number $x = 1$ is in this interval. $f(0) = 0$, $f(1) = -8$ and $f(3) = 36$. So the absolute maximum is $f(3) = 36$ and the absolute minimum is $f(1) = -8$.

II. (10) Evaluate the following limit:

$$\lim_{\theta\to 0} \frac{\cos(5\theta) - 1}{\theta^2}.$$

Solution: This is indeterminate of the form 0/0, so we can apply L'Hopital's Rule (twice):

$$\begin{aligned}\lim_{\theta\to 0} \frac{\cos(5\theta) - 1}{\theta^2} &= \lim_{\theta\to 0} \frac{-5\sin(5\theta)}{2\theta} \quad \text{(still 0/0)}\\ &= \lim_{\theta\to 0} \frac{-25\cos(5\theta)}{2}\\ &= \frac{-25}{2}.\end{aligned}$$

III. The velocity of a accelerating car is measured at each of the following times, yielding a table of values:

t = time(sec)	1	3	5	7	9	11
$v(t)$ = velocity(ft/sec)	1	3.5	5.1	7.8	9.0	11.2

A) (5) Give an estimate for the total distance traveled by the car between $t = 1$ and $t = 11$ using a *left-hand* Riemann sum for the velocity function and $\Delta t = 2$.

Solution: The left-hand Riemann sum estimate is

$$v(1)\Delta t + v(3)\Delta t + v(5)\Delta t + v(7)\Delta t + v(9)\Delta t = 2 + 7 + 10.2 + 15.6 + 18 = 52.8\text{ft}.$$

B) (5) Is your result from A) less than or greater than the actual total distance traveled, assuming v is always increasing between $t = 1$ and $t = 11$? Explain.

Solution: If v is always increasing on this interval, then each term in the left-hand Riemann sum *underestimates* the actual distance traveled on that interval. Hence the answer in A is *less* than the actual distance traveled.

IV. Compute each of the following integrals. Show all work.

A) (5) $\int_1^4 x^2 - 3\sqrt{x} + 4\ dx$

Solution: By the Evaluation Theorem, this gives:

$$\left.\frac{x^3}{3} - 2x^{3/2} + 4x\right|_1^4 = \frac{64}{3} - 16 + 16 - \frac{1}{3} + 2 - 4 = 19.$$

B) (10) $\int \sec^2(2x)\sqrt[3]{\tan(2x)+4}\ dx$

Solution: Let $u = \tan(2x) + 4$. Then $du = 2\sec^2(2x)\ dx$, and the form is

$$\frac{1}{2}\int u^{1/3}\ du = \frac{1}{2}\cdot\frac{3}{4}u^{4/3} + C = \frac{3}{8}(\tan(2x)+4)^{4/3} + C.$$

Integrate using the indicated method and the table (say which entry you are using if you use one).

C) (10) Use parts: $\int x^2 e^{4x}\ dx$ (full credit only for showing all work; you can check your result by the table)

Solution: Integrate by parts twice making u the power of x both times:

$$\begin{aligned}\int x^2e^{4x}\ dx &= \frac{x^2e^{4x}}{4} - \frac{2}{4}\int xe^{4x}\ dx\\ &= \frac{x^2e^{4x}}{4} - \frac{1}{2}\left(\frac{xe^{4x}}{4} - \frac{1}{4}\int e^{4x}\ dx\right)\\ &= \frac{x^2e^{4x}}{4} - \frac{xe^{4x}}{8} + \frac{e^{4x}}{32} + C.\end{aligned}$$

D) (12.5) Use a trigonometric substitution: $\int \frac{\sqrt{25-x^2}}{x}\ dx.$

Solution: From the form of the integrand, let $x = 5\sin\theta$, so $dx = 5\cos\theta\ d\theta$. Then the integral can be evaluated by using the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$:

$$\begin{aligned}\int \frac{\sqrt{25-x^2}}{x}\ dx &= \int \frac{5\cos\theta}{5\sin\theta}5\cos\theta\ d\theta\\ &= 5\int\frac{\cos^2\theta}{\sin\theta}\ d\theta\\ &= 5\int\frac{1-\sin^2\theta}{\sin\theta}\ d\theta\\ &= 5\int \csc\theta - \sin\theta\ d\theta\\ &= 5\left(\ln|\csc\theta - \cot\theta| + \cos\theta\right) + C \quad \#\ 15 \text{ in table}\\ &= 5\ln\left|\frac{5}{x} - \frac{\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + C.\end{aligned}$$

E) (12.5) Use partial fractions:

$$\int \frac{x+4}{(x+1)(x^2+16)}\ dx.$$

Solution: From the form of the denominator we set up the partial fractions:

$$\frac{2x+19}{(x+1)(x^2+16)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+16}$$
$$2x+19 = A(x^2+16) + (Bx+C)(x+1).$$

Equating coefficients gives $A+B=0$, $B+C=2$ and $16A+C=19$. So $A=1$, $B=-1$, and $C=3$. Integrating with #15 in table, we obtain:

$$\int \frac{dx}{x+1} + \int \frac{-x+3}{x^2+16}\ dx = \ln|x+1| - \frac{1}{2}\ln|x^2+16| + \frac{3}{4}\tan^{-1}(x/4) + C.$$

V. A) (15) A fish swimming against a current of u ft/sec (constant) for a distance L (constant) will expend energy

$$E(v) = \frac{Lv^3}{v-u}$$

if it swims at v ft/sec ($u < v$). What v minimizes the total energy expended?

Solution: By the quotient rule,

$$E'(v) = \frac{(v-u)\cdot 3Lv^2 - Lv^3\cdot(1)}{(v-u)^2},$$

and $E'(v) = 0$ when the numerator is zero, so $3L(v-u)v^2 = Lv^3$. Dividing through by Lv^2, $3(v-u) = v$, so $v = \frac{3u}{2}$. This a minimum by the first derivative test.

B) (15) Using the definition (*not* the Evaluation Theorem) compute

$$\int_2^3 x^2 + 4x\ dx.$$

The following formulas may be useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: For the right-hand Riemann sum, we have $\Delta x = \frac{3-2}{n} = \frac{1}{n}$ and $x_i = 2 + \frac{i}{n}$. Then

$$\begin{aligned}
\int_2^3 x^2 + 4x\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(2+\frac{i}{n}\right)^2 + 4\left(2+\frac{i}{n}\right)\right)\frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^2}{n^2} + \frac{8i}{n} + 12\right)\frac{1}{n} \\
&= \lim_{n\to\infty} \left(\frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{8}{n^2}\sum_{i=1}^{n} i + 12\right) \\
&= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{n^3} + \frac{8n(n+1)}{2n^2} + 12 \\
&= \frac{1}{3} + \frac{8}{2} + 12 = \frac{49}{3}.
\end{aligned}$$

(Check by Evaluation Theorem:

$$\int_2^3 x^2 + 4x \; dx = \frac{x^3}{3} + 2x^2 \Big|_2^3 = 9 + 18 - \frac{8}{3} - 8 = \frac{49}{3}.)$$