Mathematics 136, section 2 – Advanced Placement Calculus
Final Examination Solutions – December 18, 2009

I. Consider the functions:

$$y = f_1(x) = \frac{2\sin(x) + 1}{1 + x^2}, \qquad y = f_2(x) = x + x^2 + 1, \qquad y = f_3(x) = xe^{3x} + e^{x^3}$$

A) (15) Find the derivatives of f_1, f_2, f_3.

Solution: By the quotient rule,

$$f_1'(x) = \frac{2\cos(x) + 2x^2\cos(x) - 4x\sin(x) - 2x}{(1 + x^2)^2}.$$

By the derivative power rule,

$$f_2'(x) = 1 + 2x$$

By the Chain and Product rules,

$$f_3'(x) = e^{3x} + 3xe^{3x} + 3x^2e^{x^3}.$$

B) (5) Find the equation of the tangent line to the graph $y = f_1(x) + f_3(x)$ at $(0, 2)$.

Solution: $f_1(0) + f_3(0) = 1 + 1 = 2$ and $f_1'(0) + f_3'(0) = 2 + 1 = 3$. So the tangent line is $y - 2 = 3(x - 0)$, or $y = 3x + 2$.

II. Figure 1 (at the top of the next page) shows the graph $y = f(x)$ for some function $f(x)$.

A) (10) Sketch the graph $y = -f(x + 1)$.

Solution: See Figure 2 on a later page. The graph is shifted one unit left and reflected across the x-axis.

B) (10) Sketch the graph $y = f'(x)$, and explain the features of your graph, including its x-axis intercepts, its local maxima/minima, points where $f'(x)$ is undefined, and so forth.

Solution: See Figure 3 on a later page. The graph $y = f(x)$ has a local maximum at about $x = -.6$ this gives the x-intercept on the graph $y = f'(x)$. $y = f(x)$ has a cusp at $x = 0$, which produces the vertical asymptote on $y = f'(x)$. $y = f(x)$ is concave down starting at $x = -1$ to about $x = 0.5$, then concave up after that. So $y = f'(x)$ changes from decreasing to increasing at about 0.5. This gives the local minimum.

III.

A) (10) State the definition of the definite integral of a function $f(x)$ from a to b.

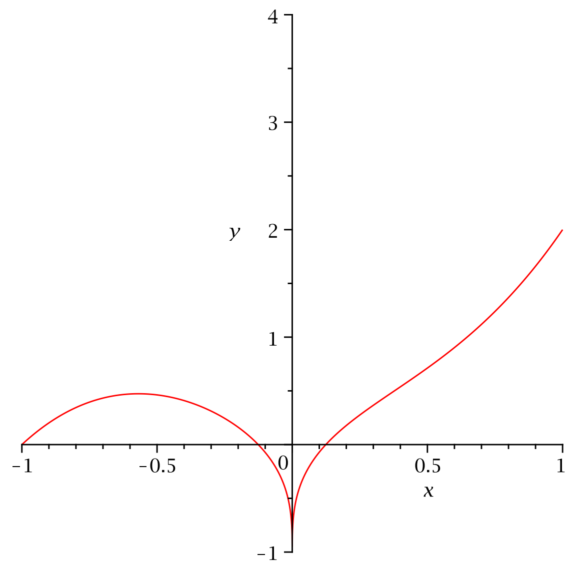

Figure 1: $y = f(x)$ in Problem II

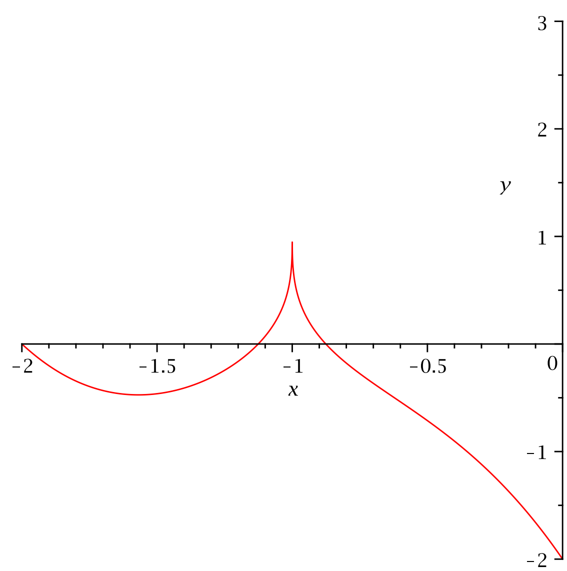

Figure 2: $y = -f(x+1)$ in Problem II

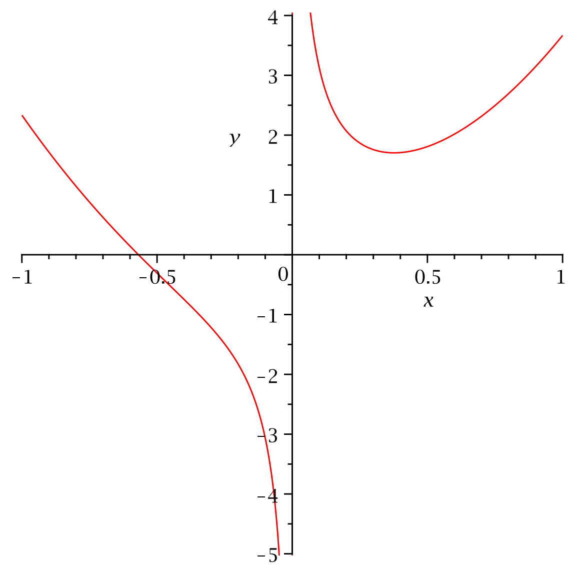

Figure 3: $y = f'(x)$ in Problem II

Solution: The definite integral of $f(x)$ over $[a, b]$ is the limit of the Riemann sums

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided that the limit exists (independently of the choice of the x_i^*).

B) (10) State both parts of the Fundamental Theorem of Calculus.

Solution: Let $f(x)$ be continuous on $[a, b]$. Then:

1. The function
$$G(x) = \int_a^x f(t)\ dt$$
is an antiderivative of $f(x)$ on (a, b) (that is $G'(x) = f(x)$ for all x in (a, b)).
2. If $F(x)$ is any antiderivative of $f(x)$, then
$$\int_a^b f(x)\ dx = F(b) - F(a).$$

IV. The region R in the xy-plane is bounded by $y = x^3 - 1$, $x = 1$, $x = 2$, and the x-axis.

A) (10) Find the volume of the solid obtained by rotating R about the x-axis.

Solution: The cross-sections by planes perpendicular to the x-axis are disks with radius $x^3 - 1$. So the volume is

$$V = \int_1^2 \pi(x^3 - 1)\ dx = \pi \int_1^2 x^6 - 2x^3 + 1\ dx = \frac{163\pi}{14}.$$

B) (5) Set up the integral to compute the volume of the solid obtained by rotating R about the line $x = 1$. You *do not* need to evaluate.

Solution: The cross-sections by planes perpendicular to the line $x = 1$ are washers with inner radius $(y+1)^{1/3} - 1$ and outer radius $2 - 1 = 1$. So the integral to compute the volume is

$$V = \int_0^7 \pi(1)^2 - \pi((y+1)^{1/3} - 1)^2\ dy.$$

C) (10) Find the coordinates of the centroid of R.

Solution: The centroid is at the point $(\overline{x}, \overline{y})$ with

$$\overline{x} = \frac{\int_1^2 x(x^3 - 1)\ dx}{\int_1^2 x^3 - 1\ dx} = \frac{94}{55}$$

and

$$\overline{y} = \frac{\int_1^2 \frac{1}{2}(x^3 - 1)^2\ dx}{\int_1^2 x^3 - 1\ dx} = \frac{163}{77}$$

V. Compute each of the following integrals. If you use an entry from the table, identify it by number.

A) (5) $\displaystyle\int x^2 \sin(3x^3)\cos(3x^3)\ dx$

Solution: Let $u = \sin(3x^3)$. Then $du = 9x^2\cos(3x^3)$ by the Chain Rule. So the form is

$$\frac{1}{9}\int u\ du = \frac{u^2}{18} + C = \frac{\sin^2(3x^3)}{18} + C.$$

B) (5) $\displaystyle\int_0^\infty \frac{dx}{x^2 + 9}$

Solution: Because of the infinite limit of integration, this is improper. We evaluate as usual:

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^2+9} &= \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+9} \\
&= \lim_{b\to\infty} \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_0^b \quad \text{using \# 17 in table} \\
&= \lim_{b\to\infty} \frac{1}{3}\tan^{-1}\left(\frac{b}{3}\right) \\
&= \frac{\pi}{6}.
\end{aligned}$$

C) (10) $\int x^2 \ln(2x)\ dx$ (use parts)

Solution: Let $u = \ln(2x)$ and $dv = x^2\ dx$. Then $du = \frac{2}{2x} = \frac{1}{x}$ and $v = \frac{x^3}{3}$. So

$$\begin{aligned}
\int x^2 \ln(2x)\ dx &= \frac{x^3\ln(2x)}{3} - \int \frac{x^3}{3x}\ dx \\
&= \frac{x^3\ln(2x)}{3} - \int \frac{x^2}{3}\ dx \\
&= \frac{x^3\ln(2x)}{3} - \frac{x^3}{9} + C.
\end{aligned}$$

(This can be checked with # 101 from the table.)

D) (15) $\int \frac{x^2}{\sqrt{25-x^2}}\ dx$ (use a trigonometric substitution)

Solution: From the form of the integrand we use the substitution $x = 5\sin\theta$ and $dx = 5\cos\theta\ d\theta$. So the integral is:

$$\begin{aligned}
\int \frac{x^2}{\sqrt{25-x^2}}\ dx &= \int \frac{25\sin^2\theta\ d\theta}{\sqrt{25(1-\sin^2\theta)}} \\
&= \int \frac{125\sin^2\theta\cos\theta\ d\theta}{5\cos\theta} \\
&= 25\int \sin^2\theta\ d\theta \\
&= \frac{25}{2}(\theta - \sin\theta\cos\theta) + C \quad \text{using \# 63 or \# 73 in the table} \\
&= \frac{25}{2}(\sin^{-1}\left(\frac{x}{5}\right) - \frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5} + C \\
&= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) - \frac{x}{2}\sqrt{25-x^2} + C.
\end{aligned}$$

VI. (15) Find the arclength of the catenary curve $y = \dfrac{e^x + e^{-x}}{2}$ from $x = 0$ to $x = 2$.

Solution: The arclength is computed by the following integration. Note that this is one of the examples where the stuff $1 + (y')^2$ under the radical is again a perfect square, after some algebra:

$$\begin{aligned}
L &= \int_0^2 \sqrt{1 + (y')^2}\, dx \\
&= \int_0^2 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx \\
&= \int_0^2 \sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}}\, dx \\
&= \int_0^2 \sqrt{\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}}\, dx \\
&= \int_0^2 \frac{e^x + e^{-x}}{2}\, dx \\
&= \left.\frac{e^x - e^{-x}}{2}\right|_0^2 \\
&= \frac{e^2 - e^{-2} - 1 + 1}{2} \\
&= \frac{e^2 - e^{-2}}{2} \doteq 3.63.
\end{aligned}$$

VII. All parts of this question refer to the differential equation

$$\frac{dy}{dx} = y^2 - y.$$

A) (10) Sketch the slope field showing the slopes at points on the lines $y = -1/2, 0, 1/2, 1, 3/2$.

Solution: See Figure 4.

B) (10) Find the solution with $y(0) = 1/2$.

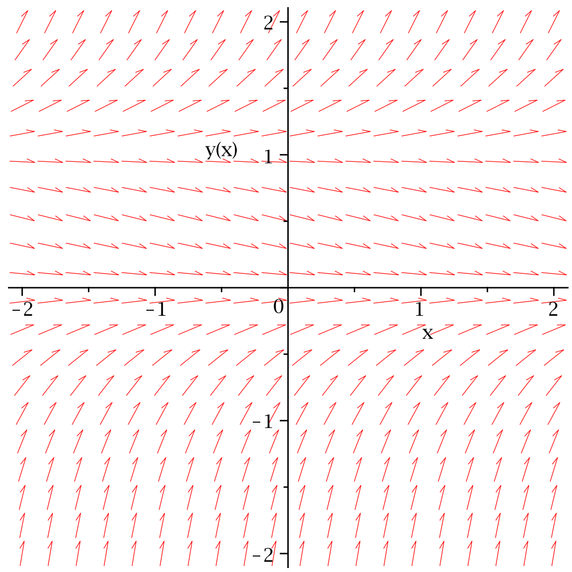

Figure 4: The slope field in VII.

Solution: We separate variables and integrate, using partial fractions on the y-integral:

$$\begin{aligned}
\int \frac{dy}{y(y-1)} &= \int dx \\
\int \frac{-1}{y} + \frac{1}{y-1}\, dy &= x + c \\
-\ln|y| + \ln|y-1| &= x + c \\
\ln\left|\frac{y-1}{y}\right| &= x + c \\
\frac{y-1}{y} &= ke^x \quad \text{where } k = \pm e^c \\
y &= \frac{1}{1-ke^x}.
\end{aligned}$$

Then from the initial condition $1/2 = \frac{1}{1-k}$. So $k = -1$, and our particular solution is $y = \frac{1}{1+e^x}$.

VIII. A hive of honey bees is attacked by a fungal infection at $t = 0$, and the population declines at a rate proportional to the *square root* of the population.

A) (5) Express this statement about the rate of growth of the population P as a differential equation.

Solution: If the population is P, then the differential equation can be written as

$$\frac{dP}{dt} = k\sqrt{P}.$$

(Since the population is declining, $k < 0$.)

B) (10) Use separation of variables to find the general solution of your differential equation.

Solution: Separating variables and integrating,

$$\begin{aligned} \int \frac{dP}{\sqrt{P}} &= \int k\ dt \\ 2P^{1/2} &= kt + c \\ P &= \left(\frac{kt+c}{2}\right)^2. \end{aligned}$$

C) (5) At $t = 0$ there were 900 bees; 441 were left after 6 weeks. When did the bee population disappear entirely?

Solution: Substituting $t = 0$ into the equation for P, we see $900 = (c/2)^2$, so $c = 60$. Then at $t = 6$, $441 = (3k + 30)^2$, so $k = -3$. The formula for the population is then $P = \left(\frac{-3t}{2} + 30\right)^2$. This equals zero when $3t/2 = 30$, so $t = 20$. After 20 weeks, the bee population has gone to zero.

IX. Series.

A) (5) Find the sum of the infinite series

$$4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \cdots.$$

Solution: This is a geometric series with $a = 4$ and $r = \frac{-2}{3}$. So the sum is

$$\frac{a}{1-r} = \frac{4}{5/3} = \frac{12}{5}.$$

B) (10) Find the Taylor series of $g(x) = e^{-x}$ at $a = 0$.

Solution: We have $g^{(n)}(x) = (-1)^n e^{-x}$, so $g^{(n)}(0) = (-1)^n$. Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots.$$

C) (10) Compute the Taylor polynomial of degree 2 for $f(x) = \sqrt{x}$ at $a = 1$ and use it to approximate $\sqrt{1.3}$. How close is the approximation?

Solution: We have $f(1) = 1$, then $f'(x) = \frac{1}{2}x^{-1/2}$, so $f'(1) = \frac{1}{2}$, and finally $f''(x) = \frac{-1}{4}x^{-3/2}$, so $f''(1) = \frac{-1}{4}$. The Taylor polynomial is

$$p_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2.$$

The approximation is

$$p_2(1.3) = 1 + \frac{1}{2}(.3) - \frac{1}{8}(.3)^2 \doteq 1.13875.$$

The calculator value is something like 1.140175. So the approximation is correct to 2 decimal places (after rounding).

Extra Credit. For a water tank with cross-section area $A(y)$ at height y above the bottom and an opening of area 1 at $y = 0$, Torricelli's Law implies that as the water drains out through the opening, the height y of the water in the tank changes according to the equation

$$A(y)\frac{dy}{dt} = -\sqrt{2gy}$$

(where g is the acceleration of gravity at the surface of the Earth).

A) (5) Suppose that the tank has the shape of the solid of revolution formed by rotating the area between a curve $x = f(y)$ and the y-axis, $0 \leq y \leq h$, about the y-axis. How should $f(y)$ be chosen in order for the water level to decrease at a *constant rate*?

   Solution: In order for the water level to be decreasing at a constant rate, we need $\frac{dy}{dt}$ constant. This will happen when $A(y)$ is proportional to $\sqrt{y}$ (note that the $\sqrt{2g}$ is constant. Now $A(y)$ is the cross-section area of the solid of revolution, so $A(y) = \pi f(y)^2$. Hence we want $f(y)^2$ proportional to $\sqrt{y}$. This says that $f(y) = cy^{1/4}$ for some constant c, or equivalently $y = kx^4$ (a surprisingly simple equation!)

B) (5) Suppose you had such a tank. Explain how you could use it as a *time-measuring device*. (Note: The ancient Egyptians actually constructed tanks of this shape and used them to measure time as early as the 16th century BC (i.e. long before calculus!). The Greeks took over the idea and their name for this sort of device was a *clepsydra*.)

   Solution: To use such a tank as a time-measurement device, the simplest approach would be to put some type of measuring stick vertically in the tank. If the stick was marked in *intervals of equal length*, then as the water dropped, the same amount of time would elapse between all pairs of consecutive marks.

Have a joyous and peaceful holiday season!