Mathematics 136, section 2 – Advanced Placement Calculus
Solutions for Exam 1 – October 1, 2009

I. All parts of this question refer to the graph $y = f(x)$ plotted in Figure 1 on the next page

A. (5) Do $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$ exist? If so, say what they are; if not say why not.

Solution: Yes, both one-sided limits exist: $\lim_{x \to 2^+} f(x) = 2$ and $\lim_{x \to 2^-} f(x) = 1/2$.

B. (5) Does $f(x)$ appear to be differentiable at $x = 3$? Why or why not?

Solution: No, there is apparently a cusp point on the graph at $x = 3$, so there is no single well-defined tangent line.

C. (10) On separate axes, plot the graphs $y = 2f(x+1)$ and $y = f(2x) + 1$. Label which graph is which.

Solution: $y = 2f(x+1)$ is shifted one unit left and stretched vertically by a factor of 2. $y = f(2x) + 1$ is compressed horizontally by a factor of 2 and shifted up one unit. See graphs in Figures 2 and 3 on the next page, and pay special attention to the scales on the two axes.

II. At time $t = 0$, a steak at temperature 0° F is taken out of the freezer and left to thaw in a kitchen heated to 70° F. The steak's temperature increases fastest at first, then at a slower and slower rate as it nears the room temperature; it never goes over 70° F.

A. (10) Sketch a graph of the temperature of the steak as a function of time. (For full credit, your graph should show the proper slopes and concavities.)

Solution: The graph should go through $(t, T) = (0, 0)$, be increasing and concave down for all $t > 0$ and tend to a horizontal asymptote at $T = 70$. Any graph showing all these properties gets full credit.

B. (5) A formula for the temperature as a function of t = time in hours is $T = 70 - 70e^{-2t}$. How long does it take for the temperature to reach 65°?

Solution: We must solve for t in $65 = 70 - 70e^{-2t}$. Isolating the e^{-2t} and taking logarithms, this gives:

$$\begin{aligned} 70e^{-2t} &= 5 \\ e^{-2t} &= \frac{1}{14} \\ -2t &= \ln\left(\frac{1}{14}\right) \\ t &= \frac{-\ln\left(\frac{1}{14}\right)}{2} \doteq 1.3 \text{ hours} . \end{aligned}$$

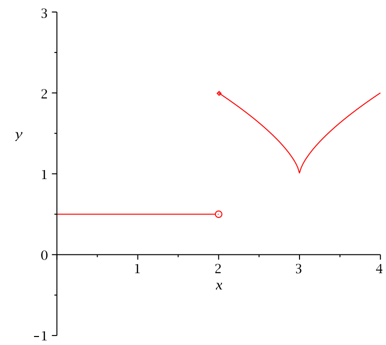

Figure 1: Figure for Problem I

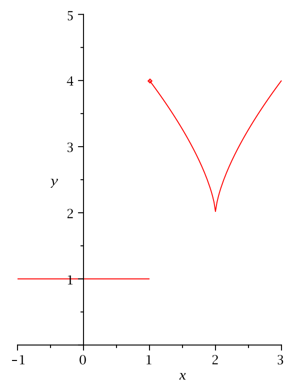

Figure 2: $y = 2f(x+1)$ in question I C

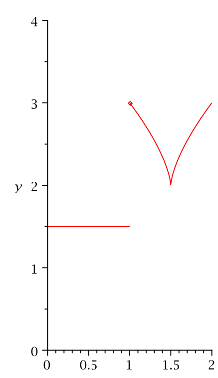

Figure 3: $y = f(2x) + 1$ in question I C

III.

A. (5) What is the exact mathematical definition of the *derivative* of a function $f(x)$ at $x = a$?

Solution: The derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided that the limit exists.

B. (10) Use the definition (not the "shortcut rules") to compute the derivative of $f(x) = \sqrt{x}$ at a general $x > 0$.

Solution: If $f(x) = \sqrt{x}$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

C. (5) What is the equation of the tangent line to $y = \sqrt{x}$ at the point $(4, 2)$?

From part B, the slope of the tangent line is $f'(4) = \frac{1}{4}$ and the equation is $y - 2 = \frac{1}{4}(x - 4)$ by the point-slope form. The equation can be rewritten as $y = \frac{1}{4}x + 1$.

IV. Compute derivatives using the "shortcut rules":

A. (5) $f(x) = \sin(x) + x^{5/3} - e^x$.

Solution: $f'(x) = \cos(x) + \frac{5}{3}x^{2/3} - e^x$.

B. (5) $g(x) = \dfrac{x^2 - x}{x^4 + 1}$

Solution: By the quotient rule,

$$\begin{aligned} g'(x) &= \frac{(x^4+1)(2x-1) - (x^2-x)(4x^3)}{(x^4+1)^2} \\ &= \frac{-2x^5 + 3x^4 + 2x - 1}{(x^4+1)^2}. \end{aligned}$$

C. (10) $h(x) = \sin^{-1}(\ln(x))$, and give the domain of $h(x)$.

Solution: By the derivative rule for inverse sine and the chain rule:

$$h'(x) = \frac{1}{\sqrt{1-(\ln(x))^2}} \cdot \frac{1}{x}.$$

The domain of $h(x)$ is the set of positive x for which $-1 \le \ln(x) \le 1$, so $e^{-1} \le x \le e$.

D. (10) Find $\frac{dy}{dx}$ by implicit differentiation if $\cos(x+y^2) + \tan^{-1}(x)y = 4$.

Using implicit differentiation (with the chain and product rules),

$$-\sin(x+y^2)(1+2yy') + \frac{y}{1+x^2} + \tan^{-1}(x)y' = 0.$$

Solving for $y' = \frac{dy}{dx}$,

$$\begin{aligned} y' &= \frac{\sin(x+y^2) - \frac{y}{1+x^2}}{\tan^{-1}(x) - 2y\sin(x+y^2)} \\ &= \frac{(1+x^2)\sin(x+y^2) - y}{(1+x^2)(\tan^{-1}(x) - 2y\sin(x+y^2))}, \end{aligned}$$

(after clearing denominators on the top).

V. (15) 100 cubic inches of cookie dough is being rolled out in a kitchen. The dough has the shape of a circular cylinder at all times, but the length of the cylinder increases and the radius decreases as times goes on. How fast is the radius of the dough cylinder changing if the length is increasing at 1 inch per second when the length is 10 inches?

Solution: The volume of a cylinder is $V = \pi r^2 h$, and both h (the "length") and r are changing with time t. The volume is constant, though. So differentiating with respect to t in the volume equation we have:

$$0 = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}. \tag{1}$$

We are given $\frac{dh}{dt} = 1$ inch per second when $h = 10$. At that time, $100 = 10\pi r^2$, so $r = \sqrt{10/\pi} \doteq 1.8$ inches. Substituting, into (1), we have

$$0 = 2\pi \cdot \sqrt{10/\pi} \cdot 10 \cdot \frac{dr}{dt} + \pi \cdot 10/\pi \cdot 1.$$

So

$$\frac{dr}{dt} = -\frac{10}{20\sqrt{10}\sqrt{\pi}} = -\frac{1}{2\sqrt{10\pi}} \doteq -.09$$

(the radius is decreasing at about .09 inch per second).

Extra Credit (10) The graph $y = e^x$ is shifted 1 unit to the left. Can the same transformed graph be obtained by a vertical stretching or shrinking of $y = e^x$? If you say yes, give the stretching or shrinking factor. If you say no, explain why not.

Solution: The answer is *yes*. Shifting to the left gives $y = e^{x+1} = e \cdot e^x$ by laws of exponents. So the shifted graph could also be obtained by stretching vertically by a factor of e(!)