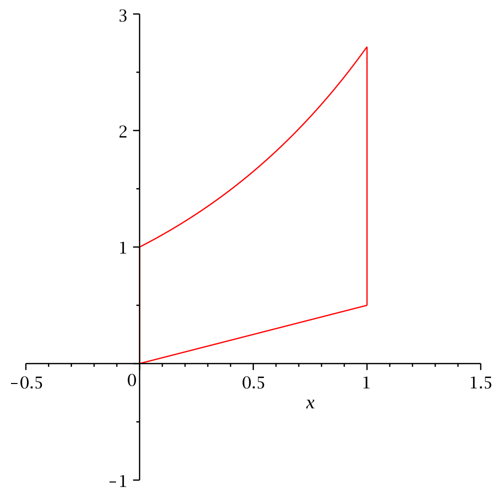

Figure 1: The region R in question I.

Mathematics 136, section 2 – Advanced Placement Calculus
Solutions for Exam 3 – December 2, 2009

I. Let R be the region in the plane bounded by $y = e^x$, $y = x/2$, $x = 0$, and $x = 1$.

A) (5) Sketch the region R.

Solution. See Figure 1.

B) (5) Find the *area* of R.

Solution: We have

$$A = \int_0^1 e^x - x/2\ dx = e^x - x^2/4\Big|_0^1 = e - 1/4 - 1 = e - 5/4 \doteq 1.47.$$

C) (10) Find the *volume* of the solid generated by rotating R about the x-axis.

Solution: The cross-sections are washers with outer radius e^x and inner radius $x/2$, so

$$\begin{aligned} V &= \int_0^1 \pi(e^x)^2 - (x/2)^2\ dx \\ &= \pi \int_0^1 e^{2x} - x^2/4\ dx \\ &= \pi \left(e^{2x}/2 - x^3/12\Big|_0^1 \right) \\ &= \pi(e^2/2 - 1/12 - 1/2) \\ &= \pi(e^2/2 - 7/12). \end{aligned}$$

D) (5) Set up, but do not evaluate, an integral or integrals to compute the *volume* of the solid generated by rotating R about the y-axis.

Solution: The cross-sections are different for different ranges of y: disks with radius $x = 2y$ for $0 \le y \le 1/2$, disks with radius 1 for $1/2 \le y \le 1$, and washers with outer radius 1 and inner radius $x = \ln(y)$ for $1 \le y \le e$. So the volume would be computed as the sum:

$$V = \int_0^{1/2} \pi(2y)^2\, dy + \int_{1/2}^{1} \pi(1)^2\, dy + \int_1^e \pi((1)^2 - (\ln(y))^2)\, dy.$$

II. A) (10) Integrate by parts: $\int xe^{-3x}\, dx$.

Solution: Let $u = x$ and $dv = e^{-3x}\, dx$. Then $du = dx$ and $v = \frac{-1}{3}e^{-3x}$. Applying the parts formula,

$$\int xe^{-3x}\, dx = \frac{-xe^{-3x}}{3} + \frac{1}{3}\int e^{-3x}\, dx = \frac{-xe^{-3x}}{3} - \frac{e^{-3x}}{9} + C.$$

B) (10) Does the integral $\int_0^\infty xe^{-3x}\, dx$ converge? If so, find its value.

Solution: The integral does converge to the value $\frac{1}{9}$:

$$\begin{aligned}\int_0^\infty xe^{-3x}\, dx &= \lim_{b\to\infty}\int_0^b xe^{-3x}\, dx \\ &= \lim_{b\to\infty}\left(\frac{-xe^{-3x}}{3} - \frac{e^{-3x}}{9}\Big|_0^b\right) \\ &= \lim_{b\to\infty}\left(\frac{-be^{-3b}}{3} - \frac{e^{-3b}}{9} + 0 + \frac{1}{9}\right).\end{aligned}$$

Now,

$$\lim_{b\to\infty}\frac{e^{-3b}}{9} = \lim_{b\to\infty}\frac{1}{9e^{3b}} = 0.$$

Moreover, by L'Hopital's Rule,

$$\begin{aligned}\lim_{b\to\infty}\frac{-be^{-3b}}{3} &= \lim_{b\to\infty}\frac{-b}{3e^{3b}} \qquad \text{(an } \infty/\infty \text{ form)} \\ &= \lim_{b\to\infty}\frac{-1}{9e^{3b}} \\ &= 0.\end{aligned}$$

Hence the integral converges to $\frac{1}{9}$.

III. (15) A water tank has the shape of an inverted cone (that is, the "point" of the cone is at the bottom) with height 10 meters and radius 3 meters. If the tank is full of water (density 1000kg per cubic meter), find the *work* done in pumping all the water out the top of the tank. The acceleration of gravity is 9.8 meters per second squared.

Solution: Consider the slice of the water, thickness Δy, originally at height y from the bottom – approximately a cylinder with radius $3y/10$. The work to lift the slice is

$$W_{slice} = \pi(3y/10)^2\Delta y \cdot 1000 \cdot 9.8 \cdot (10 - y).$$

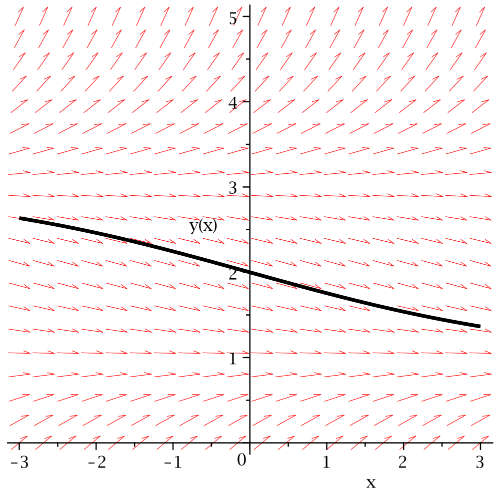

Figure 2: The slope field and solution in V.

Summing over the slices and taking a limit:

$$W = 9800\pi \int_0^{10} \frac{9y^2}{10} - \frac{9y^3}{100}\, dy = 9800\pi \left(\frac{3y^3}{10} - \frac{9y^4}{400} \Big|_0^{10} \right) = 735000\pi.$$

IV. (15) Set up and evaluate the integral to compute the *arclength* of the parabola $y = 4x^2$, $0 \le x \le 1$. (Use the table to evaluate.)

Solution: $y' = 8x$, so

$$L = \int_0^1 \sqrt{1 + (8x)^2}\, dx.$$

To evaluate, we use the substitution $u = 8x$ and # 21 in the table of integrals:

$$= \frac{1}{8} \left(\frac{8x}{2} \sqrt{1 + (8x)^2} + \frac{1}{2} \ln(8x + \sqrt{1 + (8x)^2} \Big|_0^1 \right) = \frac{\sqrt{65}}{2} + \frac{1}{16} \ln(8 + \sqrt{65}).$$

V. All parts of this question refer to the differential equation

$$\frac{dy}{dx} = (y - 1)(y - 3)/4$$

A) (10) Sketch the *slope field* of this equation, showing the slopes at points on the lines $y = 0, 1, 2, 3, 4, 5$, and $-3 \le x \le 3$.

Solution: See Figure 2.

B) (5) On your slope field, sketch the graph of the *solution* of the equation with $y(0) = 2$.

Solution: See Figure 2.

C) (10) This is a separable equation; find an explicit formula for the *solution* satisfying the initial condition $y(0) = 2$.

Solution: We separate variables and integrate using partial fractions for the y-integral:

$$\begin{aligned}
\int \frac{dy}{(y-1)(y-3)} &= \int \frac{1}{4}\, dx \\
\int \frac{-1/2}{y-1} + \frac{1/2}{y-3} &= \frac{x}{4} + c \\
-\frac{1}{2}\ln|y-1| + \frac{1}{2}\ln|y-3| &= \frac{x}{4} + c \\
\ln\left|\frac{y-3}{y-1}\right| &= \frac{x}{2} + c \\
\frac{y-3}{y-1} &= ke^{-x/2} \quad \text{where } k = \pm e^c \\
y - 3 &= k(y-1)e^{-x/2} \\
y &= \frac{-ke^{x/2} + 3}{-ke^{x/2} + 1}.
\end{aligned}$$

To get $y(0) = 2$, we must have $2 = \frac{3-k}{1-k}$ so $2 - 2k = 3 - k$, or $k = -1$. The solution can be written as

$$y = \frac{e^{x/2} + 3}{e^{x/2} + 1}.$$

Extra Credit (10) Set up the integrals to find the coordinates of the *centroid* of the region in the first quadrant inside the ellipse $\frac{x^2}{4} + y^2 = 1$. You do not need to evaluate.

The region is bounded by $y = \sqrt{1 - x^2/4}$, $x = 0$, $x = 2$, and the x-axis. The coordinates of the centroid would be computed by

$$\overline{x} = \frac{\int_0^2 x\sqrt{1 - x^2/4}\, dx}{\int_0^2 \sqrt{1 - x^2/4}\, dx}$$

and

$$\overline{y} = \frac{\int_0^2 \frac{1}{2}(\sqrt{1 - x^2/4})^2\, dx}{\int_0^2 \sqrt{1 - x^2/4}\, dx}.$$

(These are approximately $(\overline{x}, \overline{y}) \doteq (.85, .42)$. See Figure 3.)

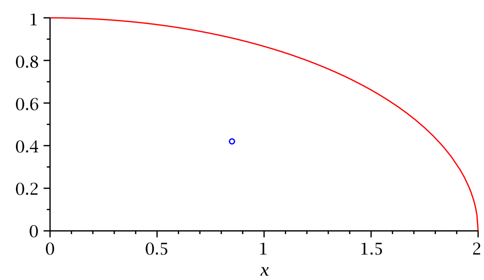

Figure 3: The region and the approximate location of the centroid.