Mathematics 373 – Applied Mathematics (PDE)
Vector Calculus – The Essentials

0. Introduction

In order to study applied problems such as heat flow in 2- and 3-dimensional regions, we will need to use *line integrals and Green's Theorem* and/or *surface integrals and the Divergence (Gauss's) Theorem.* These topics were probably either not discussed in much detail or not discussed at all in Mathematics 241 (41), Multivariable Calculus. So we will need to introduce or reintroduce the key ideas now.

1. Path Integrals, Line Integrals and Green's Theorem

Let $\alpha : [a, b] \to D$ be a smooth parametric curve in the plane. Let $T(t) = \alpha'(t)/\|\alpha'(t)\|$ be its unit tangent vector, and write $ds = \|\alpha'(t)\|\, dt$ for the arclength element on α (that is, $\int_a^b ds$ computes the arclength along the curve from $t = a$ to $t = b$. If $f(x, y)$ is a reasonably nice (e.g. continuous) *scalar* function defined on D, we can define:

Definition 1. *The* path integral *of f over α is the integral*

$$\int f ds = \int_a^b f(\alpha(t))\|\alpha'(t)\|\, dt$$

Example 1. For example, let $f = 3x + 2y$, and let $\alpha : [0, \pi] \to \mathbf{R}^2$ be defined by $\alpha(t) = (\cos(t), \sin(t))$ (the upper semicircle of the circle of radius 1 centered at $(0, 0)$). The arclength element of this curve is

$$ds = \|\alpha'(t)\| = \|(-\sin(t), \cos(t))\|\, dt = \sqrt{\sin^2(t) + \cos^2(t)}\, dt = dt,$$

Then the path integral of f over the curve α is

$$\int_\alpha f ds = \int_0^\pi 3\cos(t) + 2\sin(t)\, dt = [3\sin(t) - 2\cos(t)]_0^\pi = 4$$

A closely related concept is the *line integral of a vector field.* The best way to distinguish these in your mind is to remember that path integrals are defined for scalar functions, while line integrals are defined for vector functions (that is, vector fields).

Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ be a continuous vector field on a region $D \subset \mathbf{R}^2$. In applications, $\mathbf{F}(x, y)$ might represent a force field, the velocity field for a moving fluid, etc. Let $\alpha : [a, b] \to D$ is a smooth parametric curve whose image lies in D, with unit tangent T as above.

Definition 2. *We define the* line integral *of* $\mathbf{F}$ *along α, denoted*

$$\int_\alpha \mathbf{F} \cdot T\, ds = \int_\alpha \mathbf{F} \cdot \alpha'\, dt$$

as follows:

$$\int_\alpha \mathbf{F}\cdot T\, ds = \int_a^b \mathbf{F}(\alpha(t))\cdot \alpha'(t)\, dt$$

The notation $\mathbf{F}(\alpha(t))$ means this: If $\alpha(t) = (x(t), y(t))$ is the parametrization, and $\mathbf{F} = (P(x,y), Q(x,y))$, then $\mathbf{F}(\alpha(t)) = (P(x(t), y(t)), Q(x(t), y(t)))$. That is, we substitute the components of the parametrization for x, y in $\mathbf{F}$.

Note that the line integral of a vector field $\mathbf{F}$ could also be defined as the path integral of the scalar function $\mathbf{F}\cdot T$ (which can be interpreted as *the component of* $\mathbf{F}$ *in the direction of* T, the unit tangent to the curve).

In physics, if $\mathbf{F}$ represents a *force* applied to an object moving along the path α, then $\int_\alpha \mathbf{F}\cdot T\, ds$ represents the *total work done* (the integral of the component of $\mathbf{F}$ in the direction of the path).

For computing line integrals, we usually use the observation that the factors of $\|\alpha'(t)\|$ in the denominator from T, and in the numerator from ds always cancel. Hence there is no need to compute T or ds, and we almost always simply compute $\int_a^b \mathbf{F}(\alpha(t))\cdot\alpha'(t)\, dt$.

Example 2. Let α be the curve from Example 1. If $\mathbf{F}$ is the vector field $\mathbf{F} = (x^2, y+1)$, then the line integral of $\mathbf{F}$ along α is equal to

$$\begin{aligned}
\int_0^\pi \mathbf{F}(\alpha(t))\cdot\alpha'(t)\, dt &= \int_0^\pi (\cos^2(t), \sin(t)+1)\cdot(-\sin(t), \cos(t))\, dt \\
&= \int_0^\pi -\cos^2(t)\sin(t) + \sin(t)\cos(t) + \cos(t)\, dt \\
&= [\cos^3(t)/3 - \cos^2(t)/2 + \sin(t)]_0^\pi \\
&= -2/3
\end{aligned}$$

The most important theorem about line integrals deals with the case where α is a *simple closed curve* – that is, a smooth curve α where $\alpha(a) = \alpha(b)$ but α is one-to-one otherwise on the domain $[a, b]$. This means, intuitively, that α does not cross itself, except that the final point $\alpha(b)$ is the same as $\alpha(a)$. A simple closed curve divides the plane into an interior and an exterior region. (This fact seems intuitively obvious but is actually quite difficult to prove rigorously – it is the statement of a famous result called the Jordan Curve Theorem.) Green's Theorem gives an equality between the line integral of a vector field over a simple closed curve and a related double integral over the interior region.

Green's Theorem. *Let* $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ *be a vector field whose components have continuous first order partial derivatives on an open* $U \subset \mathbf{R}^2$ *containing the simple closed curve* α *and the region* D *interior to* α. *If* α *is traced so that the region* D *is always to the left as we proceed along* α, *then*

$$\int_\alpha \mathbf{F}\cdot T\, ds = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA \tag{1}$$

(where dA represents the area element in the plane). If α is traced in the other direction, then a minus sign must be inserted on one side of this equation.

Example 3. For instance, let $\mathbf{F} = (-y, x)$, and let $\alpha = (5\cos(t), \sin(t))$, $t \in [0, 2\pi]$. Here α is an ellipse (semi-major axis $= 5$, semi-minor axis $= 1$, center at $(0, 0)$, traced counterclockwise). Hence the condition that α is traced so that the region D is always to the left *is satisfied.* Let us verify that the conclusion (1) of Green's Theorem is true here. The line integral evaluates to

$$\int_\alpha \mathbf{F} \cdot T\, ds = \int_0^{2\pi} 5\sin^2(t) + 5\cos^2(t)\, dt = 10\pi$$

The double integral is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{-5}^{5} \int_{-\sqrt{1-x^2/25}}^{\sqrt{1-x^2/25}} 2\, dy\, dx$$

which equals 2 times the area of the ellipse. Recalling the area formula $A = \pi ab$ for an ellipse with semi-major axis a and semi-minor axis b, this equals $2 \cdot \pi \cdot 5 \cdot 1 = 10\pi$, which agrees with the line integral calculation.

Green's Theorem can be used in several ways. First, the relation given in the statement is an important step in many proofs about line and double integrals in the plane. Second, to compute a line integral around a simple closed curve, it may sometimes be simpler to apply Green's Theorem and compute $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$ instead.

Example 4. For instance, consider the vector field

$$\mathbf{F} = (P, Q) = (x^2 + y, x + y^3)$$

Note that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0$. Hence, *the line integral of* $\mathbf{F}$ *around any simple closed curve is* **zero**, by Green's Theorem. In fact, note $\mathbf{F}$ is a gradient vector field: $\mathbf{F} = \nabla f$, where $f = x^3/3 + xy + y^4/4$. If $\mathbf{F} = \nabla f$ is any gradient vector field, then Green's Theorem implies

$$\int_\alpha \nabla f \cdot T ds = \iint_D \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\, dA$$

If f has continuous second order partial derivatives, then the mixed second order partials are *equal*, so the integrand is zero and the line integral is zero.

In fact, even without using Green's Theorem, it can be seen that:

Proposition. *Let* $\mathbf{F}$ *be the gradient vector field of a function* f *with continuous first order partial derivatives on* U. *Then the line integral of* $\mathbf{F}$ *around every simple closed curve in* U *is zero.*

Proof: By the Chain Rule for partial derivatives,

$$\int_\alpha \mathbf{F} \cdot T\, ds = \int_\alpha \nabla f \cdot T\, ds = \int_a^b \nabla f(\alpha) \cdot \alpha'(t)\, dt = \int_a^b \frac{d}{dt} f(\alpha(t)) dt = f(\alpha(b)) - f(\alpha(a))$$

Since α is a simple closed curve the endpoints are the same, and the integral is zero.//

It also follows by the same reasoning that in this case, the line integral of $\mathbf{F}$ *depends only on the endpoints of the path of integration*, not on the actual curve α. $\mathbf{F}$ is said to be a *conservative* vector field if $\mathbf{F} = \nabla f$ for some scalar function f. If $\mathbf{F}$ is conservative, then f is called a *potential function* for $\mathbf{F}$.

There is also a partial converse of the proposition, which states that if $\mathbf{F}$ is continuous on a *simply connected* region (intuitively, a region with "no holes") and the line integrals of $\mathbf{F}$ around all simple closed curves are zero, then $\mathbf{F}$ is a conservative vector field: $\mathbf{F} = \nabla f$ for some potential function f.

2. Surface Integrals and the Divergence (Gauss's) Theorem

Corresponding to line integrals over curves in the plane and Green's Theorem, we have integrals over surfaces in $\mathbf{R}^3$ and a result called the Divergence (or Gauss's) Theorem. This theorem asserts the equality of the surface integral of a vector field over a closed surface and a triple integral over the volume enclosed.

Let $\Phi : [a, b] \times [c, d] \to \mathbf{R}^3$ be a smooth parametric surface in $\mathbf{R}^3$. Then we can write $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$. For instance:

1) The graph $z = f(x, y)$ can be parametrized as the image of $\Phi(u, v) = (u, v, f(u, v))$, where (u, v) ranges over the domain of f.
2) The plane in $\mathbf{R}^3$ through the origin spanned by vectors (a, b, c) and (d, e, f) can be parametrized as

$$\Phi(u, v) = u(a, b, c) + v(d, e, f) = (au + dv, bu + ev, cu + fv)$$

3) Another important surface parametrization is the spherical coordinate parametrization for the unit sphere:

$$\Phi(\theta, \varphi) = (\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi))$$

for $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi]$.

A parametric surface contains two families of coordinate curves – the u-curves obtained by setting v equal to a constant and letting u vary, and the v-curves obtained by setting u equal to a constant and letting v vary. At each point of the surface, the vector

$$\frac{\partial \Phi}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

is tangent to the u-curve through that point. Similarly,

$$\frac{\partial \Phi}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

is tangent to the v-curve through that point. Hence the *tangent plane* to the surface at $\Phi(u_0, v_0)$ is the plane containing that point on the surface and the two vectors $\frac{\partial \Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial \Phi}{\partial v}(u_0, v_0)$. The *unit normal* to the surface is the unit vector perpendicular to the tangent plane obtained via:

$$\mathbf{N} = \frac{\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}}{\|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\|}$$

For example, the unit normal to a graph $\Phi(u, v) = (u, v, f(u, v))$ is

$$\mathbf{N} = \frac{\left(1, 0, \frac{\partial f}{\partial u}\right) \times \left(1, 0, \frac{\partial f}{\partial u}\right)}{\| \left(1, 0, \frac{\partial f}{\partial u}\right) \times \left(1, 0, \frac{\partial f}{\partial u}\right) \|} = \frac{\left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)}{\sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}}$$

The surface area element of a parametric surface is

$$dA = \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| \, dudv$$

Integrating dA over a portion of the surface gives the surface area. (Intuitively, the idea is that when we integrate dA we are summing up the areas of infinitesimal parallelograms spanned by the tangent vectors to the u- and v-curves at each point. Recall that the area of the parallelogram spanned by two vectors $\mathbf{A}$ and $\mathbf{B}$ in 3-space is $\|\mathbf{A} \times \mathbf{B}\|$.)

If $\mathbf{F}$ is a vector field on some $U \subset \mathbf{R}^3$ containing the image of a parametrization mapping Φ, then we can define the surface integral of $\mathbf{F}$ over Φ as follows.

Definition 3. *The surface integral of* $\mathbf{F}$ *over* Φ, *denoted* $\iint \mathbf{F} \cdot \mathbf{N} \, dA$, *is defined to be*

$$\int_a^b \int_c^d \mathbf{F}(\Phi(u, v)) \cdot \mathbf{N}(u, v) \, dA = \int_a^b \int_c^d \mathbf{F}(\Phi(u, v)) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) \, dvdu.$$

As in the case of line integrals, the norm factors from the unit normal and the area element *always cancel*. Physically, surface integrals have interpretations like this: If $\mathbf{F}$ is the velocity vector field for a moving fluid in a 3D region, then $\iint_\Phi \mathbf{F} \cdot \mathbf{N} \, dA$ represents the total flow of fluid across the surface.

Example 5. For instance, let us compute the surface integral of $\mathbf{F} = (x, 0, 0)$ over the unit sphere in the spherical coordinate parametrization given above:

$$\Phi(\theta, \varphi) = (\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi))$$

We begin by computing the normal vector:

$$\begin{aligned}
\frac{\partial \Phi}{\partial \theta} &= (-\sin(\theta)\sin(\varphi), \cos(\theta)\sin(\varphi), 0) \\
\frac{\partial \Phi}{\partial \varphi} &= (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), -\sin(\varphi)) \\
\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial \varphi} &= (-\sin^2(\varphi)\cos(\theta), -\sin^2(\varphi)\sin(\theta), -\sin(\varphi)\cos(\varphi))
\end{aligned}$$

Hence

$$\begin{aligned}\iint_\Phi \mathbf{F}\cdot\mathbf{N}\,dA &= \int_0^{2\pi}\int_0^{\pi} (\sin(\varphi)\cos(\theta),0,0)\cdot\left(\frac{\partial\Phi}{\partial\theta}\times\frac{\partial\Phi}{\partial\varphi}\right)\,d\varphi d\theta\\ &= \int_0^{2\pi}\int_0^{\pi} -\sin^3(\varphi)\cos^2(\theta)\,d\varphi d\theta\\ &= -4\pi/3\end{aligned}$$

If Φ describes a simple closed surface, then that surface divides $\mathbf{R}^3$ into an exterior volume and an interior volume. Depending on the parametrization, the vector $\mathbf{N}$ could end up pointing inward or outward. By the properties of the cross product, though, if we wanted an outward unit normal and $\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}$ pointed inward instead, then we could simply reverse the order of the factors to get an outward normal: $\frac{\partial\Phi}{\partial v}\times\frac{\partial\Phi}{\partial u}$.

If $\mathbf{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is a vector field on some region in $\mathbf{R}^3$ with differentialble component functions, then we define the *divergence* of $\mathbf{F}$, written $\nabla\cdot\mathbf{F}$ or $\operatorname{div}(\mathbf{F})$ to be the scalar function:

$$\nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

Then the Divergence (Gauss's) Theorem states:

Divergence Theorem. *Let $\mathbf{F}$ be a vector field whose components have continuous first order partial derivatives on an open set $U\subset\mathbf{R}^3$ containing a simple closed surface Φ and the volume D inside. Let $\mathbf{N}$ be the outward unit normal to Φ. Then*

$$\iint_\Phi \mathbf{F}\cdot\mathbf{N}\,dA = \iiint_D \nabla\cdot\mathbf{F}\,dV \tag{2}$$

(If $\mathbf{N}$ is the inward normal, then a minus sign must be inserted on one side to give an equality.)

Example 6. For instance, consider the vector field and surface from Example 5. It is easy to see that the $\mathbf{N}$ we computed there is actually the *inward* normal to the sphere. On the other hand

$$\nabla\cdot\mathbf{F} = \frac{\partial x}{\partial x}+0+0 = 1.$$

Hence the volume integral computes the volume inside the unit sphere:

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = 4\pi/3$$

(from the volume formula for a sphere of radius $r=1$. Hence the conclusion (2) of the Divergence Theorem does hold after inserting the extra minus sign:

$$\iint_\Phi \mathbf{F}\cdot\mathbf{N}\,dA = -4/3 = -\iiint_D \nabla\cdot\mathbf{F}\,dA,$$

3. An Alternate Form of Green's Theorem

Using our notation, this result reads as follows: If α is a simple closed curve, then

$$\int_\alpha \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA, \tag{3}$$

where $\mathbf{N}$ is now the outward unit normal vector to the curve α, and $\nabla \cdot \mathbf{F}$ is the "2D-divergence" of $\mathbf{F}$: If $\mathbf{F} = (P(x, y), Q(x, y))$, then

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Let's see the connection between (3) and the original statement of Green's Theorem above, (1).

First, we note that if $\alpha(t) = (x(t), y(t))$, then the vector $(y'(t), -x'(t))$ is orthogonal to the tangent vector $\alpha'(t)$. (It is obtained by rotating $\alpha'(t)$ by $\pi/2$ clockwise.) Hence as our unit normal we can take

$$\mathbf{N} = \frac{(y'(t), -x'(t))}{\|(y'(t), -x'(t))\|} = \frac{(y'(t), -x'(t))}{\|\alpha'(t)\|}.$$

(Whether $\mathbf{N}$ points outward or inward for a simple closed curve depends on the parametrization we started from.)

Then, using this $\mathbf{N}$, we have

$$\int_\alpha \mathbf{F} \cdot \mathbf{N}\, ds = \int_a^b P(x(t), y(t))y'(t) - Q(x(t), y(t))x(t)\, dt$$

Note that this is the same thing as we would get for the line integral of

$$\mathbf{G} = (-Q(x, y), P(x, y))$$

around α. But then by the usual form (1) of Green's Theorem, if α is traversed so that D lies to the left, we have:

$$\int_\alpha \mathbf{F} \cdot \mathbf{N}\, ds = \int_\alpha \mathbf{G} \cdot T\, ds = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA = \iint_D \nabla \cdot \mathbf{F}\, dA$$