Mathematics 373 – Applied Mathematics (PDE)
Problem Set 4
March 27, 2001

I. For each of the following vector fields and surfaces, verify the Divergence Theorem by direct computation of the surface integral and the triple integral:

A) $\mathbf{F} = (z, y, x)$, S the sphere of radius 1 centered at $(0, 0, 0)$.

B) $\mathbf{F} = (x + y, y + z, x + z)$, S the surface of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$. (For the surface integral, you will need to parametrize each of the four faces of the tetrahedron separately, and add the corresponding surface integrals.) *Answer: 1/2 (both values).*

II. Recall the definitions of the hyperbolic cosine and sine functions:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

A) Show that for all $x \in \mathbf{R}$, $\cosh^2 x - \sinh^2 x = 1$. Consider the parametric curve $\alpha(t) = (\cosh t, \sinh t)$ in the plane. How does this curve help explain the names *hyperbolic cosine* and *sine*?

B) Express the derivatives of the hyperbolic functions $\cosh x, \sinh x$, and the functions $\tanh x = \frac{\sinh x}{\cosh x}, \coth x = \frac{\cosh x}{\sinh x}, \mathrm{sech} x = \frac{1}{\cosh x}, \mathrm{csch} x = \frac{1}{\sinh x}$ analogous to the other trig functions in terms of the same hyperbolic functions.

C) Show that any function $g(x) = Ae^x + Be^{-x}$ can be written as a linear combination of $\cosh(x)$ and $\sinh(x)$.

II. Consider the heat equation

$$u_t = k(u_{xx} + u_{yy})$$

on the rectangle $[0, a] \times [0, b]$ in the (x, y)-plane. (Think of a thin metal plate in this shape and heat conduction within the plate.) An *equilibrium temperature distribution* is a solution $u(x, y, t)$ of the heat equation that *does not depend on* t – in other words, a solution that has "settled down" and is not changing any longer. When u is independent of t, the heat equation reduces to

$$u_{xx} + u_{yy} = 0 \tag{1}$$

which is called *Laplace's Equation*. In this problem, we will apply separation of variables and Fourier series techniques to solve a boundary value problem for Laplace's Equation on the rectangle.

A) Say $u(x, y) = X(x)Y(y)$ solves (1). What 2 ODE's are satisfied by X and Y?

B) A typical problem here would be to solve (1) together with boundary conditions such as

$$u_y(x, 0) = u_y(x, b) = 0,$$

all x (horizontal edges of the plate insulated), and

$$u(0, y) = 0 \qquad u(a, y) = g(y)$$

for some given function g (vertical edges held at given temperatures). What are the eigenfunctions and eigenvalues for the boundary value problem you get for $Y(y)$?

C) Show how to develop a Fourier series solution of Laplace's Equation satisfying the boundary conditions from B and these initial conditions. (Note: you will probably want to use the hyperbolic functions from problem II here to deal with the factors $X(x)$ in the product solutions.)

IV. Suppose we have a thin metal ring with constant c, ρ, K_0 in the shape of $D = \{(r, \theta) : a \le r \le b, 0 \le \theta \le 2\pi\}$ in the polar coordinate plane. The ring has a radially symmetric equilibrium temperature function $u = u(r)$ that satisfies

$$\frac{\partial u}{\partial r}(a) = \beta, \quad \frac{\partial u}{\partial r}(b) = 1.$$

Show that $\beta = b/a$ and interpret this result physically. (Note: As for the case of rectangular coordinates, $\frac{\partial u}{\partial r}(a)$ is proportional to the heat flux across the inner boundary of the ring. Similarly at $r = b$.)

Assignment

Individual writeups due in class, Tuesday, April 3.