Mathematics 373 – Applied Mathematics (PDE)
Problem Set 3
March 15, 2001

I. The first step in our proof of the Dirichlet pointwise convergence theorem for Fourier series was the rather "magical" formula for the partial sum of the Fourier series of a function f:

$$s_N(x) = \frac{1}{L}\int_{-L}^{L} f(x+u)D_N(u)\, du, \tag{1}$$

where D_N is the *Dirichlet kernel function*:

$$D_N(u) = \frac{1}{2} + \sum_{n=1}^{N} \cos\left(\frac{n\pi u}{L}\right).$$

In this problem, we want to see that this really works, at least in a particular example(!) You'll probably want to use Maple for all parts here.

A) Plot $D_N(u)$ for $[-L, L] = [-1, 1]$, for $N = 5, 10, 20$. What seems to be happening as N increases?

B) Using Maple, construct and plot a function giving the part of the periodic extension of

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \end{cases}$$

for $-3 \le x \le 3$. Call your function f. Now use the following commands to construct a function as in (1) above with $N = 5$ and plot it:

```
D5:=u->1/2+add(cos(n*Pi*u),n=1..5);
PS:=x->evalf(Int(f(x+u)*D5(u),u=-1..1));
plot(PS,-2..2);
```

Warning: This is a rather large computation because integrals of rapidly oscillating functions are difficult by numerical methods. Expect the plot to take about 10 minutes to display.

C) Now compute the 5th partial sum of the Fourier series for this function on $[-1, 1]$ directly from the integral formulas for the Fourier coefficients (as in Lab/Demo 1), and plot that. Are your results the same?

D) What happens if you only use f on $[-1, 1]$ and not the periodic extension in B above? Explain.

II. Evaluate the line integrals of each of the following vector fields along the indicated curves, and answer the accompanying questions.

A) $\mathbf{F} = (x, x + y)$, $\alpha(t) = (t - t^2, t^2 + t^3)$, $t \in [0, 1]$. *Answer: 139/60.*

B) $\mathbf{F} = (y^2, 2xy)$, α the path from $(-1,2)$ to $(1,5)$ made up of two straight line segments, the first parallel to the x-axis, the second parallel to the y-axis.
C) $\mathbf{F}$ = same vector field as in part B, but now α the line segment directly from $(-1,2)$ to $(1,5)$.
D) Your answers for B and C should be equal. Explain why this is true. *Hint:* Is $\mathbf{F}$ a gradient vector field?
E) The vector field

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

α the circle of radius 1 centered at $(0,0)$ in $\mathbf{R}^2$, traversed counterclockwise.

III. Use Green's Theorem to evaluate each of the following line integrals around the specified simple closed curve, and answer the other questions.

A) $\mathbf{F} = (\sin(\pi xy), \cos(\pi xy))$, α the positively-oriented boundary of the square $[0,1] \times [0,1]$. *Answer:* $-\frac{2+\pi}{\pi}$.
B) $\mathbf{F} = (y^2, xy)$, α the ellipse with equation $x^2/9 + y^2/4 = 1$.
C) Compute

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

for the vector field $\mathbf{F}$ and the region D inside α from part E of question II. Your answer should be different from what you saw in that part. Is this a contradiction to Green's Theorem?? Explain.

IV. The statement of Green's Theorem is also valid in more general situations. For example, consider the *circular ring* D in the plane between the two circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$. The positively oriented boundary of D consists of the the circle of radius 2 traversed counterclockwise, and the circle of radius 1 traversed clockwise. Let $\mathbf{F} = (P, Q)$ be a vector field whose component functions have continous first order partial derivatives on some open set in $\mathbf{R}^2$ containing the whole ring. Show that the line integral of $\mathbf{F}$ over the positively oriented boundary is equal to the double integral of $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ over the ring. *Hint: Consider the curve α_1 which consists of the line segment from $(1,0)$ to $(2,0)$, followed by the upper semicircle of radius 2 traversed counterclockwise, followed the line segment from $(-2,0)$ to $(-1,0)$, followed by the upper semicircle of radius 1 traversed clockwise. Similarly let α_2 be the curve consisting of the line segment from $(-1,0)$ to $(-2,0)$, followed by the lower semicircle of radius 2 traversed counterclockwise, followed by the line segment from $(2,0)$ to $(1,0)$, followed by the lower semicircle of radius 1 traversed clockwise.*

Assignment

Individual solutions, due Thursday, March 22.