Mathematics 373 – Applied Mathematics
Individual Problem Set 2 – D'Alembert's Solution of the Wave Equation
Due: Tuesday February 6, 2001

A) Solve $u_{tt} = c^2 u_{xx}$ for $x \in \mathbf{R}$, $u(x,0) = e^x$, $u_t(x,0) = \sin(x)$. Sketch (or use Maple to plot) the solution (u versus x, with t fixed) for $t = 0, 1, 2, 3$. What are the two travelling waves that are superposed to get the solution $u(x,t)$?

B) A string originally at rest with $u(x,0) = f(x) = 0$ is struck with a hammer at time $t = 0$. The hammer blow imparts an initial velocity

$$u_t(x,0) = g(x) = \begin{cases} 1 & \text{if } |x| \le a \\ 0 & \text{if } |x| > a. \end{cases}$$

1) Show that the solution of $u_{tt} = c^2 u_{xx}$ with these initial conditions is

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds,$$

and explain why this says $u(x,t)$ is $\frac{1}{2c}$ times the length of the interval $(x-ct, x+ct) \cap (-a, a)$ for all t, x.
2) Plot u versus x (t fixed) for $t = \frac{a}{2c}, \frac{a}{c}, \frac{3a}{2c}, \frac{2a}{c}, \frac{5a}{2c}, \frac{5a}{c}$, and describe what the string is doing in words.
3) As a function of t, what is the maximum displacement $\max_{x \in \mathbf{R}} u(x,t)$?

C) Recall that we derived D'Alembert's solution of the wave equation by factoring the operator $\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$. Apply the same method to solve

1) $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$.
2) $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, f, g arbitrary.