Mathematics 373 – Applied Mathematics (PDE)
Solution for Problem C, PS 1

The problem was to solve the Cauchy problem

$$\begin{cases} u_x + 2u_y + u = e^{3x-2y} \\ u(x,0) = 0 \end{cases}$$

The solution of this inhomogeneous linear 1st order PDE has the form $u = u^p + u^h$, where u^h is the general solution of the associated homogeneous equation $u_x + 2u_y + u = 0$, and u^p is any particular solution.

We find u^h by changing variables:

$$\begin{cases} r = 2x - y \\ s = x + 2y \end{cases}$$

The homogeneous equation becomes $5u_s + u = 0$, so

$$u = g(r)e^{-s/5} = g(2x-y)e^{-(x+2y)/5}$$

where g is an arbitrary function of one variable.

To find u^p, we can use the idea of undetermined coefficients. e^{3x-2y} is a solution of the homogeneous equation, so we must try a function such as $u^p = A(3x-2y)e^{3x-2y}$. Then

$$\begin{aligned} u^p_x + 2u^p_y + u^p &= e^{3x-2y}(3A + 3A(3x-2y) - 4A - 4A(3x-2y) + A(3x-2y)) \\ &= -Ae^{3x-2y} = e^{3x-2y} \end{aligned}$$

when $A = -1$. This gives a general solution

$$u = g(2x-y)e^{-(x+2y)/5} - (3x-2y)e^{3x-2y}.$$

Finally, we want $u(x,0) = 0$, so

$$0 = g(2x)e^{-x/5} - 3xe^{3x} \Rightarrow g(2x) = 3xe^{16x/5}$$

This says

$$g(x) = \frac{3}{2}xe^{8x/5}$$

so

$$\begin{aligned} u &= \frac{3}{2}(2x-y)e^{(16x-8y)/5}e^{(-x+2y)/5} - (3x-2y)e^{3x-2y} \\ &= \left(3x - \frac{3y}{2}\right)e^{3x-2y} - (3x-2y)e^{3x-2y} \\ &= \frac{y}{2}e^{3x-2y} \end{aligned}$$