Mathematics 373 – Applied Mathematics (PDE)
Individual Problem Set 1 – First Order Linear PDE
January 18, 2001

A. (Review from Multivariable Calculus) Recall that the chain rule from 1-variable calculus says that if g is differentiable at x and f is differentiable at $g(x)$ then the composition $f(g(x))$ is differentiable at x and

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

A first generalization of this to functions of several variables is the following: If $f(x,y)$ is a differentiable function of two variables and $\alpha(t) = (x(t), y(t))$ is a differentiable parametric curve with image in the domain of f, then

$$(1) \qquad \frac{d}{dt} f(\alpha(t)) = \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$

where ∇f is the gradient vector of f.

1) Use (1) to compute $\frac{d}{dt} f(\alpha(t))$ if $f(x,y) = x^2 - y^2 + x^3$ and $\alpha(t) = (t^2 - 1, t^3 - t)$. What does your result say about the curve α and f?
2) If $\alpha(t) = (x_0, y_0) + t(u, v)$ where (u, v) is a unit vector,

$$\frac{d}{dt} f(\alpha(t))|_{t=0} = \nabla f(\alpha(0)) \cdot \alpha'(0) = \nabla f(x_0, y_0) \cdot (u, v)$$

is the *directional derivative* of f at $\alpha(0) = (x_0, y_0)$ in the direction of the vector $\alpha'(0) = (u, v)$. Fixing the point (x_0, y_0), consider the directional derivative as a function of the direction vector (u, v). In which direction is the directional derivative *maximized*? In which direction is the directional derivative *minimized*? In which directions is the directional derivative *equal to zero*? Prove your assertions.
3) The *line integral* of a vector field $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ along a parametric curve $\alpha(t) = (x(t), y(t))$ $(a \le t \le b)$ in the plane is *defined to be*

$$\int_\alpha \mathbf{F} = \int_a^b \mathbf{F}(\alpha(t)) \cdot \alpha'(t)\, dt$$

Show that if $\mathbf{F} = \nabla f$ for some $f(x,y)$, then the line integral doesn't actually depend on the path α, only on the starting and final points of the path: $\alpha(a)$ and $\alpha(b)$.

B. Solve the PDE $4u_x + 7u_t = 0$ with the Cauchy condition $u(x, 0) = \sin(3x) - \cos(2x)$.

C. Use the change of coordinates method to solve $u_x + 2u_y + u = e^{3x-2y}$ with $u(x, 0) = 0$.

D. Solve $(1 + x^2)u_x + u_y = 0$ for $u(x,y)$. Sketch some of the characteristic curves.

E. Solve $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. In which region of the plane is the solution uniquely determined?

F. What happens if we try to solve a first order linear PDE $a(x, y)u_x + b(x, y)u_y = 0$ with the condition $u(x, 0) = f(x)$, but the x-axis *is a characteristic curve* of the equation? (Note: There are two cases!)

Assignment

Individual write-ups due in class on Thursday, January 25.