Mathematics 373 – Applied Mathematics
Introductory Topics in PDE
January 13, 2001

§*1. What is a PDE and where do they come from?*

Last semester we studied the beginnings of the theory of *ordinary differential equations.* This semester, we turn our attention to the study of *partial differential equations.* Like an ordinary differential equation, a partial differential equation (PDE) is a relation between derivatives of an unknown function (the "dependent variable"). Only now the function depends on several independent variables and other known functions of those variables, and the derivatives are partial derivatives. If the independent variables are $(x_1, \ldots, x_n)$, the relation is assumed to be true on some set in $\mathbf{R}^n$ (possibly for all $(x_1, \ldots, x_n)$).

To simplify notation in writing PDE, we will usually indicate the derivatives with subscripts. For instance, $\frac{\partial u}{\partial x}$ will be written as u_x. Higher order partial derivatives are indicated the same way. For example, if $u = u(x, y)$, then u_{xxx} means $\frac{\partial^3 u}{\partial x^3}$ and u_{xy} means $\frac{\partial^2 u}{\partial x \partial y}$. Some first examples of PDE are:

1) $u_x + 5u_t + u = \sin(x)\cos(t)$, where $u = u(x, t)$,

2) $u_t = k u_{xx}$, where $u = u(x, t)$ again,

3) $u_{xx} + u_{yy} + u_{zz} = \sin(u)$, where $u = u(x, y, z)$,

4) $u_t + u u_x + u_{xxx} = 0$, where $u = u(x, t)$,

5) $u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + u_{\theta\theta}\right)$, where $u = u(r, \theta, t)$.

As was the case last semester for ODE, we will say a PDE is *linear* if the unknown function and its derivatives only appear linearly in the equation. Of the examples above, 1,2,5 are linear PDE (the c^2/r multiplying the u_r term in equation 5 is OK – the equation is still linear – since the $1/r$ does not depend on the dependent variable u). But 3 and 4 are not (because of the $\sin(u)$ term in 3 and the uu_x term in 4). Examples 2 and 5 are *homogeneous linear*, while 1 is an *inhomogeneous linear* equation.

The *order* of a PDE is the highest order derivative that appears (this is also the same idea as for ODE). For instance, in the examples above, 1 is a first order equation, 2, 3, 5 are second order equations, and 4 is a third order PDE.

Most of the PDE that we will study come from models of physical, chemical, or biological systems. A typical situation that leads to PDE is when we are studying a quantity in a system that depends *both on time and on the location in space* where the quantity is measured. For example, we will see shortly that equations such as 2 above arise in the study of heat conduction along a long thin wire (when the thermal properties of the wire are constant along its whole length). Since the temperature will usually depend both on the location along the wire (measured by the *spatial* coordinate x) where the measurement is made, *and* on the time when the measurement is made, we naturally have to deal with a function $u(x, t)$ of two variables to describe the temperature of the wire completely. In a similar fashion, we will see that equation 5 gives a model for the vibrations of a circular membrane like a drumhead under certain physical assumptions. Here there are two spatial coordinates, r, θ specifying the location of a point on the drumhead in *polar coordinates* – the natural choice of coordinates for a circular geometry – and time t as a third independent variable. The function $u(r, \theta, t)$ in this case will represent the *vertical displacement* of the membrane at the location (r, θ) and at time t.

Most of our work this semester will be devoted to understanding different techniques of solving PDE, possibly together with other side conditions such as specified initial conditions, or specified values for u at particular values of spatial coordinates. We will begin with the general class of linear first order PDE in 2 variables. These already illustrate some of the key patterns concerning PDE and the differences between PDE and ODE. We will not try to provide a complete treatment of every possible special case. (For several reasons to be discussed below, that is very difficult if not impossible in the most naive sense – i.e. if we are looking primarily for analytic formulas for solutions.) Instead we will give a rapid survey focusing on a few important types of these equations.

§*2. First order linear PDE*

The general first order linear PDE for $u = u(x, t)$ has the form:

$$a(x, t)u_x + b(x, t)u_t + c(x, t)u = d(x, t).$$

(Even though we write t here for the second independent variable, we are not necessarily thinking of t as representing time. Both variables could represent spatial coordinates in equations of this form.) We will begin our study of these equations by considering *homogeneous equations* ($d(x, t) \equiv 0$), in which $c(x, t) \equiv 0$ as well, and in which a, b are constants. In this case, it will be easy to derive the form of solutions. Moreover, the features of this special case are representative of what happens in general.

For instance, let's consider the PDE

$$3u_x + 2u_t = 0. \tag{1}$$

To understand what (1) says about the function $u(x, t)$, and use that understanding to derive the form of solutions, we need to recall some facts from multivariable calculus. First, the partial derivatives u_x, u_t are the components of the *gradient vector*

$$\nabla u(x, t) = (u_x(x, t), u_t(x, t)) = \left(\frac{\partial u}{\partial x}(x, t), \frac{\partial u}{\partial t}(x, t)\right).$$

Thinking of the operations of vector algebra, we see that the equation (1) just says that the vector $(3, 2)$ has *dot product* 0 with ∇u: $0 = (3, 2) \cdot (u_x, u_t) = 3u_x + 2u_t$. Hence, using the geometric interpretation of the dot product, the vector $(3, 2)$ is perpendicular to ∇u at every point. The next ingredient we need is the geometric meaning of the gradient vector of a function. Recall that the gradient ∇u is perpendicular to the contours or level curves of the function u at every point. Since $(3, 2) \perp \nabla u$ and $\nabla u \perp$ the level curves of u (and since we are working in the 2-dimensional (u,t)-plane), we see: $(3,2)$ must be tangent to the level curves of u at every point. A moment's thought shows that this means that *the level curves of the function* $u(x,t)$ *are the lines with* $(3,2)$ *as a direction vector*, which are all the lines of the form $2x-3t=c$, c constant. (Note: we chose $(2,-3)$ as a convenient normal vector since its components contain no fractions. Any other normal vector such as $(1,-3/2)$ would work just as well, and the level curves could also be written as $x-3t/2=c$.) Hence $u(x,t)$ must take constant values along each of the lines $2x-3t=c$. This means that u is a function of $2x-3t$ only. In other words, every solution of (1) must have the form $u(x,t)=f(2x-3t)$ for some function f of one variable.

Conversely, now, suppose $u(x,t)=f(2x-3t)$ for some differentiable one variable function f. Then by the chain rule:

$$\begin{aligned} u_x &= f'(2x-3t)\cdot\frac{\partial}{\partial x}(2x-3t)=2f'(2x-3t)\\ u_t &= f'(2x-3t)\cdot\frac{\partial}{\partial t}(2x-3t)=-3f'(2x-3t)\end{aligned}$$

As a result,

$$3u_x+2u_t=3\cdot 2f'(2x-3t)+2\cdot(-3f'(2x-3t))=0$$

so the PDE (1) is satisfied.

In other words, we have shown that the the solutions of (1) are exactly the functions $u(x,t)=f(2x-3t)$, where f is an *arbitrary differentiable function of one variable*. This is the first big difference between PDE and ODE. Recall we saw last semester that the general solution of a first order ODE has one arbitrary *constant*. The general solution of a first order PDE contains an *arbitrary function of one variable* instead. This means that PDE typically have a *much larger* collection of solutions than ODE! In fact, in the collection of solutions of (1), we can find a particular solution agreeing with any given function at points of a *curve that intersects each level curve* $2x-3t=c$ *only once*.

For instance, suppose we wanted to solve (1) together with the side (initial) condition $u(x,0)=\sin(x)$. The line $t=0$ (the x-axis in the (x,t)-plane) meets each level curve of the solutions in just one point. Hence in the general solution $u(x,t)=f(2x-3t)$ of (1), if we set $t=0$ we see that $f(2x)=\sin(x)$, so $f(x)=\sin(x/2)$. Therefore the solution of (1) and $u(x,0)=\sin(x)$ is the function

$$u(x,t)=\sin((2x-3t)/2)=\sin(x-3t/2).$$

For equations $au_x+bu_t=0$ with a,b constant (and not both zero, or else the equation is pretty uninteresting!), the situation is the same as in the example we just worked out. The geometric reasoning we applied shows that the general solution has the form $u(x,t)=f(bx-at)$ where f is an arbitrary differentiable function of one variable. Solutions are constant along the lines $bx-at=c$, c constant.

We will also now present a second, more algebraic, way to derive this fact. The algebraic method will also be useful for more general first order linear PDE with constant coefficients on the u_x,u_t terms. As we know from the geometric approach, the lines $bx-$

$at = c$ are important for this problem. So let's consider changing coordinates in the (x, t) plane, introducing

$$
\begin{aligned}
r &= bx - at \\
s &= ax + bt
\end{aligned}
\tag{2}
$$

as new rectangular coordinates, where a, b are the constant coefficients appearing in the PDE. We choose s to have this form since this makes the lines $s = const$ perpendicular to the lines $r = const$, as is the case for the original x, t-coordinate system.

If u is any function of r, s, then we can also think of u as a function of x, t via (2). Conversely, since we assume $(a, b) \neq (0, 0)$, the equations (2) can also be solved for x, t as functions of r, s:

$$
\begin{aligned}
x &= \frac{1}{a^2 + b^2}(br + as) \\
t &= \frac{1}{a^2 + b^2}(-ar + bs)
\end{aligned}
\tag{3}
$$

and we can think of any function of r, s as a function of x, t.

Now we see what becomes of the PDE $au_x + bu_t = 0$ in the new r, s coordinate system. By the chain rule and (2),

$$
\begin{aligned}
u_x &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = bu_r + au_s \\
u_t &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = -au_r + bu_s
\end{aligned}
$$

Hence

$$0 = au_x + bu_t = a(bu_r + au_s) + b(-au_r + bu_s) = (a^2 + b^2)u_s \Rightarrow u_s = 0.$$

This says that if we think of u as a function of r, s, then u depends *only on r, not on s*. Therefore, $u(r, s) = f(r) = f(bx - at)$ for some function f of one variable, as we expect.

We ask, what does the form of this solution mean? First note that every function $f(bx - at)$ of $bx - at$ alone can also be seen as a function $g(x - \frac{at}{b})$ of $x - \frac{at}{b}$ alone (just rescale to get $g(x) = f(bx)$). Even though we do not always think of t as representing time, a particularly nice picture emerges if we do think that way now. At $t = 0$ we have $u(x, 0) = g(x)$, so $g(x)$ gives the initial shape of the solution as function of t. But then for other t, $u(x, t) = g(x - \frac{at}{b})$, which just gives a shifted copy of the graph of g. Depending on the signs of a, b, the shifting is either to the left or to the right (left if $\frac{at}{b} < 0$ and right if $\frac{at}{b} > 0$). The amount of shifting depends linearly on t. Hence the solutions of equations of this form are all "traveling waves" with a constant shape that propagates to the left or right as t increases. First order linear equations with constant coefficients are therefore models for what is called *transport*, e.g. motion at a constant velocity.

We next turn to the case where the coefficients of u_x and u_t are not constant. In this case, the geometric meaning of the PDE

$$a(x, t)u_x + b(x, t)u_t = 0 \tag{4}$$

is the same: At each point (x,t), the vector $(a(x,t),b(x,t))$ is perpendicular to the gradient ∇u, hence tangent to the level curves of u. Since (a,b) is not a constant vector, though, the level curves are usually *not straight lines*. Instead, they form a family of curves called the *characteristic curves* of the PDE. We can find the characteristics and the general solution of the PDE by the following:

Method of Characteristics. *To find the general solution of*

$$a(x,t)u_x + b(x,t)u_t = 0,$$

1) *Solve the following ODE:*

$$\frac{dx}{dt} = \frac{a(x,t)}{b(x,t)}$$

(If it is more convenient to do so, it is also possible think of x as the independent variable in this equation and solve

$$\frac{dt}{dx} = \frac{b(x,t)}{a(x,t)}.)$$

2) *Write the solutions in the implicit form $\phi(x,t) = c$.*
3) *Then the general solution of (4) will have the form*

$$u(x,t) = f(\phi(x,t))$$

where f is an arbitrary differentiable function of one variable.

Here is an example illustrating how this process works. We look for the general solution of the first order linear PDE

$$(x-t)u_x + tu_t = 0 \tag{5}$$

The equation of the characteristic curves is the ODE

$$\frac{dx}{dt} = \frac{x-t}{t} = \frac{1}{t}x - 1$$

which is a first order linear equation with $p = \frac{-1}{t}$ and $q = -1$. The general solution of this ODE is

$$x = ct - t\int \frac{1}{t}\,dt = ct - t\ln|t|$$

Writing this in the required implicit form, we get

$$\phi(x,t) = \frac{x + t\ln|t|}{t} = \frac{x}{t} + \ln|t| = c.$$

This is the equation of the characteristic curves of (5). So the general solution of (5) looks like

$$u(x,t) = f\left(\frac{x}{t} + \ln|t|\right)$$

for an arbitrary differentiable function f of one variable. (Note that no solution is defined at $t = 0$.)

As in the constant coefficient case we discussed above, particular solutions taking specified values at the points of "non-characteristic" curves (curves intersecting characteristics only once) can be found from the general solution by solving for f. For instance, the line $t = 2$ intersects the characteristic $\frac{x}{t} + \ln|t| = c$ only at $x = (c - \ln(2))/2$. Hence a particular solution of (5) satisfying the condition $u(x,2) = e^x$ (for instance) can be determined by substituting $t = 2$ in the general solution. We get

$$e^x = f(x/2 + \ln(2))$$

Hence $f(x) = e^{2x - 2\ln(2)} = e^{2x}/4$, and our desired particular solution is

$$u(x,t) = e^{\frac{2x}{t} + 2\ln|t|}/4 = \frac{1}{4}t^2 e^{\frac{2x}{t}}.$$

As you can probably see, being able to produce an analytic formula for $u(x,t)$ depends on being able to solve the ODE defining the characteristic curves. It is very easy to write down examples where that equation has no solutions in elementary functions. In those cases the method fails in the sense that we cannot use it to get a formula (unless we define new "special functions" to solve the problem). *However, it is also very important to understand the underlying geometry of the method of characteristics from a conceptual standpoint. That is always valid even if the method cannot be followed to yield a formula for the general solution of the PDE. In other words, the method of characteristics is as much a way to understand what the solutions of first order linear PDE look like conceptually and graphically as it is an algorithm to produce analytic solutions.*

We will conclude this section with two additional examples illustrating how some first order linear equations involving u terms and inhomogeneous terms can be solved by the methods we have introduced. We will not consider the most general equations of these forms, which frequently have no elementary solutions. Instead, we will focus on the linear equations with constant coefficients.

Our first example is the general constant coefficient homogeneous linear PDE:

$$au_x + bu_t + cu = 0 \tag{6}$$

To solve this, we use the change of coordinates from (2) above:

$$\begin{aligned} r &= bx - at \\ s &= ax + bt. \end{aligned}$$

As before, we have

$$\begin{aligned} u_x &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = bu_r + au_s \\ u_t &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = -au_r + bu_s \end{aligned}$$

So the equation (6) transforms to

$$a(bu_r + au_s) + b(-au_r + bu_s) + cu = (a^2 + b^2)u_s + cu = 0$$

in the new coordinate system. For PDE like this one involving a derivative with respect to only one of the independent variables, we can solve using methods from ODE. This is essentially just a separable ODE for u, but the arbitrary "constant" will now be a function of the other variable r (this is how the arbitrary function enters the solution in this case). We have

$$u_s = \frac{-c}{a^2 + b^2} u$$

so

$$u(r, s) = g(r) e^{\frac{-cs}{a^2+b^2}},$$

where g is an arbitrary differentiable function of one variable. In terms of the original x, t variables, this becomes:

$$u(x,t) = g(bx - at) e^{\frac{-c(ax+bt)}{a^2+b^2}} \tag{7}$$

Caution: In this case the lines $bx - at = const$ are not level curves for the solution. Because of the other exponential factor $u(x,t)$ is not a function of $r = bx - at$ alone. Thus, even though many authors call the lines $bx - at = const$ characteristic curves here too, the characteristics are not the level curves for equations of this form.

Finally, we consider an example of this type where the equation is not homogeneous:

$$u_x + 2u_t + 3u = e^{x-4t}.$$

As we saw for linear ODE last semester, it is also true for PDE that the general solution of an inhomogeneous equation can be written as

$$u = u_h + u_p$$

where the subscripts here do *not* stand for partial derivatives, but rather for "homogenenous" and "particular": u_h is the general solution of the corresponding homogeneous equation with right side equal to zero, and u_p is any particular solution of the equation. (The proof of this assertion is the same as for ODE: Since the equation is linear, if u and u_p are any two solutions, then $u - u_p$ solves the corresponding homogeneous equation.)

We find the general solution of the homogeneous equation as in the solution (7) of the general equation (6). We have $a = 1, b = 2, c = 3$, so

$$u_h(x,t) = g(2x - t) e^{\frac{-3(x+2t)}{1^2+2^2}} = g(2x - t) e^{\frac{-3x-6t}{5}}.$$

To find a particular solution, we can use a variant of the method of undetermined coefficients from ODE. Be aware that this is possible only because of the special form of the right hand side of this equation, though. By the chain rule, we have

$$\frac{\partial}{\partial x} e^{x-4t} = e^{x-4t} \qquad \frac{\partial}{\partial t} e^{x-t} = -4e^{x-4t}$$

Hence we see that a particular solution of the form

$$u_p = Ae^{x-4t}$$

will exist. We must choose A so that $A - 4A + A = -2A = 1$, or $A = -1/2$. Therefore we get a general solution

$$u(x,t) = g(2x-t)e^{\frac{-3x-6t}{5}} - \frac{1}{2}e^{x-4t}.$$

A particular solution satisfying the initial condition $u(x,0) = 0$ (for instance) can be found from the equation:

$$0 = u(x,0) = g(2x)e^{-3x/5} - \frac{1}{2}e^{x}$$

Hence

$$g(2x) = \frac{1}{2}e^{x} \cdot e^{3x/5} = \frac{1}{2}e^{8x/5}$$

so $g(x) = \frac{1}{2}e^{4x/5}$.

§3. D'Alembert's solution of the wave equation with one spatial dimension

In §10.2 of Nagle, Saff, and Snider, the wave equation with one space variable is given in the form

$$u_{tt} = c^2 u_{xx}$$

where $c > 0$ is a constant. We will begin this section by (very) briefly indicating how this PDE is derived. We consider a long, thin, flexible, elastic, wire or string (think of a piano or guitar string) made of a material that is uniform along the length of the string. In the problems we will solve later, we will often assume that the ends of the string are fixed. If the string is plucked and released, then the tension each segment of the string exerts on its neighboring segments will cause the string to move back towards its rest position at all times. Hence the string will tend to oscillate or vibrate around its rest position.

We will assume that the string vibrates only in a single plane. (If the plucked position of the string is planar, an actual string will tend to vibrate in just one plane, at least to a good approximation.) We will use an ordinary rectangular xy coordinate system in this plane, where the rest position of the string is on the x-axis ($y = 0$). If we assume that the string is stretched taut to begin with, then the shape of the string will be given by some graph $y = u(x,t)$ at all later times. In other words the string is only vibrating "up and down" (transversely) and not "from side to side" (longitudinally) in the plane. The function $u(x,t)$ can also be understood as the *vertical displacement function* – it gives the difference between the actual location of the point on the string with coordinate x at time t and the rest position of that point.

Because of our assumptions on the flexibility of the string, if we look at any finite segment, say from $x = x_1$ to x, then the tension forces exerted by the rest of the string on the segment will be represented by vectors $\mathbf{T}(x_1,t)$ and $\mathbf{T}(x,t)$ that are *tangent* to the string at $x = x_1$ and x respectively (for all t). Let $T(x,t)$ be the magnitude of the tension force at x at time t. Because the string is not vibrating longitudinally, only the vertical component of the tension is producing motion. At each point the vertical component of the tension is $T\sin(\theta)$, where θ is the angle between the horizontal and the tangent to the string. We have $\sin(\theta) = u_x/\sqrt{1+u_x^2}$. But if we assume that the vertical displacement is *small compared to the length of the string*, then the values of u_x will also be small, and we can use the *approximation*

$$\sqrt{1+u_x^2} \doteq 1.$$

Then Newton's Law $F = ma$ implies that

$$\int_{x_1}^{x} \rho u_{tt}\, dx = T u_x|_{x_1}^{x}$$

where ρ is the mass density of the string (a constant). (We ignore the approximation made above and take this as an exact relation.) Differentiating both sides with respect to x gives $\rho u_{tt} = T u_{xx}$ or

$$u_{tt} = c^2 u_{xx}$$

where $c = \sqrt{\frac{T}{\rho}}$. This is the wave equation with one space variable.

Our next goal is to derive some first information about the solutions of this second order linear PDE. A physically meaningful problem to be solved would be one where we have a string of finite length, L say. We can take the string to correspond to the interval $[0, L]$ on the x-axis. Moreover, we might assume that the ends of the string are fixed at 0 displacement, and the initial shape of the string and the initial velocity at each point of the string are known. Mathematically, this translates into an *initial-boundary value problem*:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(0,t) &= u(L,t) = 0, \text{ all } t \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x). \end{aligned} \tag{8}$$

In most cases here, we would look for solutions defined for $x \in [0, L]$ and not care whether the definition could be extended beyond that interval. We will see how to bring the "boundary conditions" $u(0,t) = u(L,t) = 0$ for all t into the picture in a few days. For now, let us see how our work with first order linear ODE can be applied to solve the wave equation without the assumptions about the range of realistic values for x. In other words, let us consider solutions of the wave equation and initial conditions as in (8) but ignoring the boundary conditions. The solutions we get will include the solutions of the PDE plus the boundary conditions, so we will get the solutions we want plus many others. Interestingly, though, if the string is very long and we are not near one of the ends (x is far from 0 and L), then we could think of every solution as representing an approximation to a possible behavior of the string at that x. The same results we will see are also derived by a slightly different method in [NSS] on pages 654-5.

The derivation we will see is based on the following observation. The wave equation is a second order linear homogeneous equation that can be written in the "operator form"

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u = 0 \tag{9}$$

Because the coefficients are constant here and because the mixed partials of a nice (C^2) function satisfy $u_{xt} = u_{tx}$, this can be rewritten in the following form by factoring the second order wave equation operator into a product of two first order partial differential operators. The equation (9) is equivalent to

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0 \tag{10}$$

To make use of this, let $v = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u$ Then (10) says v satisfies the first order linear PDE with constant coefficients

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v = cv_x + v_t = 0$$

The results of §2 say that the general solution of this equation is

$$v(x,t) = \phi(x - ct)$$

where ϕ is an arbitrary differentiable function of one variable. As we saw there, we can think of v as a traveling wave of constant shape propagating to the right ($c > 0$) at a constant speed c.

Next, the wave equation (9) is equivalent to the equation

$$u_t - cu_x = v(x,t)$$

where $v(x,t) = \phi(x - ct)$ is the function just found. This is an inhomogeneous first order linear PDE. The general solution can be found by the method discussed in the example at the end of §2. The general solution of the associated homogeneous equation is $u(x,t) = B(x + ct)$ where B is an arbitrary differentiable function. Note that this represents a second traveling wave of constant shape, propagating to the left ($c > 0$) as t increases. To find a particular solution, we note that if $u(x,t) = A(x - ct)$ is any function of $x - ct$ alone, then by the chain rule

$$u_t - cu_x = -cA'(x - ct) - cA'(x - ct) = -2cA'(x - ct).$$

Hence, given ϕ as above, if $A = \frac{-1}{2c}\int \phi(s)\, ds$ (that is $\frac{-1}{2c}$ times an antiderivative of ϕ), then $u(x,t) = A(x - ct)$ solves $u_t - cu_x = \phi(x - ct)$. Since ϕ was general, though, so is A. Our conclusion is that any function of the form

$$A(x - ct) + B(x + ct) \tag{11}$$

solves the wave equation (9). Conversely, it is easy to check that every function of this form (A,B any two arbitrary differentiable functions) satisfies the wave equation.

In other words, *the general solution of the wave equation on* $\mathbf{R}$ *looks like a superposition of two traveling waves, one propagating to the left at constant speed* c*, one propagating to the right at constant speed* c. The shapes of the two waves are completely arbitrary and need not be the same. This result was discovered by the French mathematician and philosopher D'Alembert in the 18th century.

Continuing from (11) it can be seen that the solution of the wave equation and the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ (f,g given) can be written as

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds.$$

The two traveling waves are given by $A(x-ct)$ and $B(x+ct)$ for

$$A(x) = \frac{1}{2}\left(f(x) - \frac{1}{2c}\int_0^x g(s)\ ds\right)$$

and

$$B(x) = \frac{1}{2}\left(f(x) + \frac{1}{2c}\int_0^x g(s)\ ds\right)$$

See page 655 in [NSS] or the class notes for a proof.