Mathematics 373 – Applied Mathematics (PDE)
Selected Solutions – Midterm Problem Set
March 12, 2001

I. B. As always, the value of a solution of a first order linear PDE with prescribed values along some curve in the plane is only uniquely determined on the subset of the plane consisting of all points on characteristics *that intersect* the curve where the values are prescribed. Here the characteristics are the curves $y = \arcsin(x) + c$ for all real c. With the condition $u(x, 0) = \sqrt{1 - x^2}$, we have to determine which characteristics intersect the x-axis (the given curve). By considering the graph of the arcsin function, we see that only the characteristics with $-\pi/2 \le c \le \pi/2$ intersect the x-axis here. (If $c < -\pi/2$, the graph is shifted "too far down" along the y-axis; if $c > \pi/2$ it is shifted "too far up".) Hence the region where the solution is uniquely determined is

$$\{(x, y) : -1 \le x \le 1, \arcsin(x) - \pi/2 \le y \le \arcsin(x) + \pi/2\}$$

II. A. Using D'Alembert's solution of the wave equation:

$$u(x,t) = \frac{1}{2}(f(x - ct) + f(x + ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds$$

we assume that if $f(x), g(x) = 0$ for all $|x| > B$. Then $|x| > B+ct$ is the same as $x < -(B+ct)$ or $x > B+ct$. From this we get $x+ct < -B$ (which implies $x-ct = (x+ct)-2ct < -B$ too since we take $t \ge 0$) or $x - ct > B$ (which implies $x + ct = (x - ct) + 2ct > B$ too). Therefore $f(x - ct) = f(x + ct) = 0$. Similarly the interval $[x - ct, x + ct] \subset (-\infty, -B]$ in the first case, and $[x - ct, x + ct] \subset [B, \infty)$ in the second case. Therefore the interval of integration is contained in the range where g is known to be zero. Hence $\int_{x-ct}^{x+ct} g(s)\ ds = 0$.

II. B. We have

$$KE + PE = \int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2\ dx$$

Hence by the hint and the wave equation:

$$\begin{aligned}\frac{d}{dt}(KE + PE) &= \frac{d}{dt}\int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2\ dx \\ &= \int_{-\infty}^{\infty} 2\rho u_t u_{tt} + 2T u_x u_{xt}\ dx \\ &= 2T\int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt}\ dx\end{aligned}$$

Integrating by parts in the first term ($U = u_t, dV = u_x\ dx$), we get

$$\begin{aligned}&= 2T u_t u_x|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} -u_{xt}u_x + u_x u_{xt}\ dx \\ &= 2T u_t u_x|_{-\infty}^{\infty} + 0\end{aligned}$$

Now, by part A of the problem, if $|x| > B + ct$, we have $u(x,t) = 0$. Hence the limit as $x \to \pm\infty$ of u or any of its derivatives must be zero. Hence

$$\frac{d}{dt}(KE + PE) = 0$$

which is what we wanted to show.

IV. B. The eigenvalues and eigenfunctions of the boundary value problem

$$\begin{cases} X''(x) = -\lambda X(x) \\ X(0) = 0 \\ X'(L) = 0 \end{cases}$$

come only from the case $\lambda > 0$, so $X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. The condition $X(0) = 0$ implies $A = 0$. Then the other boundary condition says $B\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$. So $\sqrt{\lambda}L = \frac{(2n-1)\pi}{2}$ for some $n \geq 1$ (that is, $\sqrt{\lambda}L$ has the form (odd integer)$\cdot\pi/2$).

eigenvalues: $\lambda = \left(\frac{(2n-1)\pi}{2L}\right)^2$.
eigenfunctions: $\phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$.

IV B. To derive the orthogonality relation

$$0 = \int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right)\sin\left(\frac{(2m-1)\pi x}{2L}\right)\,dx$$

when $m \neq n$, use the trig identity

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

then integrate.

V. A. The Fourier sine coefficients of the given initial string shape are

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\left[\int_0^a \frac{hx}{a}\sin\left(\frac{n\pi x}{L}\right)\,dx + \int_a^L \frac{h(L-x)}{(L-a)}\sin\left(\frac{n\pi x}{L}\right)\,dx\right] \\ &= \frac{2hL^2}{a(L-a)}\cdot\sin\left(\frac{n\pi a}{L}\right)\cdot\frac{1}{(n\pi)^2} \end{aligned}$$

The solution of the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$