Mathematics 373 – Applied Mathematics (PDE)
Midterm Problem Set
February 22, 2001

Ground Rules

This is an open book, open notes, open library, open Maple, open SW 335, unlimited time exam. If you find something of use in a library book, please footnote it (this is not necessary for topics covered in our course textbook). You must work independently, and you may not consult other faculty members in working on these problems. *When you have completed the problem set, please write a brief statement saying that you have abided by these ground rules, sign it, and turn it in with your solutions.*

Due: No later than 5:00 pm, Thursday, March 1.

I. Consider the 1st order linear PDE for $u(x,y)$:

$$\sqrt{1-x^2}\,u_x + u_y = 0 \tag{1}$$

A) Find the solution of (1) with $u(0,y) = \frac{1}{y^2+4}$.
B) What is the largest set in the (x,y)-plane on which the value of a solution of (1) together with $u(x,0) = \sqrt{1-x^2}$ is *uniquely* determined? Find the solution.

II. Consider an infinite string satisfying the wave equation

$$\rho u_{tt} = T u_{xx}, \tag{2}$$

where ρ (the mass density), and T (the tension) are assumed constant. We consider solutions of (2) together with the initial conditions:

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x). \end{cases}$$

We will *assume that* $f(x)$ and $g(x)$ are zero for all x with $|x| > B$ for some $B > 0$. In this problem, we will see why solutions of (2) automatically satisfy the physically important property of *conservation of energy.*

A) Show that $u(x,t) = 0$ for all $|x| > B + ct$, where $c = \sqrt{\frac{T}{\rho}}$ as in our derivation of the wave equation.
B) The total energy of the string is the sum of a *kinetic energy* term and a *potential energy* term. The kinetic energy KE is the total $\frac{1}{2}mv^2$ over the whole string, or $KE = \int_{-\infty}^{\infty} \rho u_t^2\, dx$. The potential energy is, by definition: $PE = \int_{-\infty}^{\infty} T u_x^2\, dx$. Both KE and PE are typically nonconstant functions of t. However, show using the wave equation that $\frac{d}{dt}(KE + PE) = 0$ (i.e. total energy is constant). Hint: You may

freely assume that derivatives with respect to t and integrals with respect to x may be interchanged; in other words: $\frac{d}{dt}\int v(x,t)\,dx = \int v_t(x,t)\,dx$ is legal.

III. Show that the orthogonality relation

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases} 1 & \text{if } m = n \neq 0 \\ 2 & \text{if } m = n = 0 \\ 0 & \text{if } m \neq n \end{cases}$$

holds.

IV. Consider the following "mixed " boundary value problem for the heat equation:

$$(4) \qquad \begin{cases} u_t = ku_{xx} & \text{(heat equation)} \\ u(x,0) = f(x) & \text{(given initial temperature)} \\ u(0,t) = 0 & \text{all } t \text{ (one end of wire held at zero temperature)} \\ u_x(L,t) = 0 & \text{all } t \text{ (other end of wire } \textit{insulated}\text{)} \end{cases}$$

A) What are the 2 ODE's we get by separating variables?

B) What are the eigenvalues and eigenfunctions for the equation for $X(x)$? Note that the conditions here are $X(0) = X'(L) = 0$.

C) You should be able to index your eigenvalues by a positive integer $n \geq 1$. Call the corresponding eigenfunction ϕ_n. Show that in this case too we have an orthogonality relation:

$$\int_0^L \phi_m(x)\phi_n(x)\,dx = 0$$

if $m \neq n$.

V. *The Plucked String.* Consider the following boundary value problem for the wave equation:

$$(3) \qquad \begin{cases} u_{tt} = c^2 u_{xx} & & \text{(wave equation)} \\ u(x,0) = \begin{cases} hx/a & \text{if } 0 \leq x \leq a \\ h(L-x)/(L-a) & \text{if } a \leq x \leq L \end{cases} & & \\ u_t(x,0) = 0 & & \text{(0 initial velocity)} \\ u(0,t) = u(L,t) = 0 & & \text{(ends fixed at zero displacement)} \end{cases}$$

A) Using your results from question C on Discussion 2 giving the eigenvalues and eigenfunctions for problems of this form, derive a formal Fourier series solution of (3). (Notes: The Fourier expansion for the initial string position will come by considering the extension of $f(x)$ to the interval $[-L, L]$ as an *odd* function. It will contain only $\sin\left(\frac{n\pi x}{L}\right)$ terms.)

B) Take $c = 1$, $h = .5$, $L = 10$, $a = 2$ and plot partial sum approximations to your solution in Maple. (Extra Credit for a Maple animation of the solution – see the online Help for technical hints. If you want to go for this option, please email me a copy of the .mws file you develop to do the animation so I can check that it works properly!)