Mathematics 373 – Applied Mathematics (PDE)
Lab Day on Bessel Functions
April 19, 2001

Background

Last time, we introduced the Bessel functions of the first kind via the series expansions:

$$J_m(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell\,!\,(m+\ell)\,!}\left(\frac{z}{2}\right)^{2\ell+m} \tag{1}$$

Using the Ratio Test, we saw that the series for $J_m(z)$ in (1) converges for all $z \in \mathbf{R}$, so it gives a well-defined function on $\mathbf{R}$. (Actually, it converges for *complex* z as well, but that is another story!) At this point in your acquaintance with them, though, Bessel functions probably seem like "magic" since apart from the name $J_m(z)$ we have only the description (1). Today, we want to get more familiar with these functions and their properties using Maple to generate plots, work with some of the formulas, etc.

Maple Details

The Bessel functions $J_m(z)$ are built-in in Maple. The Maple notation for this function is `BesselJ(m,z)`. You can think of `BesselJ(m,z)` as a function of two variables – the first (namely the m) is the *order*, the second is the actual *argument*, z. Since these functions are built-in, you can use them in a Maple session just like the elementary functions *sin, cos, tan, exp, ln, arctan* and so forth. For instance you can operate on Bessel functions with other Maple commands, you can plot them or functions made using them and other functions, etc.

Lab Questions, part 1

A) You can recover the series expansion (1) for $J_m(z)$ in the case that m is a positive integer using the `taylor` command in Maple. Try this command:

```
taylor(BesselJ(2,z),z=0,11);
```

The output is the terms of degree < 11 in the Taylor series of $J_2(x)$ at $x = 0$. Does this agree with (1) above? Check using Maple's `sum` command to write out the same terms as in (1) with $m = 2$:

```
sum((-1)^l/(l!*(2+l)!)*(z/2)^(2*l+2),l=0..10);
```

B) Plot $J_0(z), J_1(z), J_2(z), J_3(z)$ together on the same set of axes, with $z \in [0, 10]$. Which function is which and how can you tell? Note: In (1), the term of lowest degree (in z) in the series for $J_m(z)$ is z^m. What does that tell you about the shape of the graph $y = J_m(z)$ for z near zero?

C) The Bessel functions and their derivatives satisfy a plethora of different relationships. For example, try entering the command

```
diff(BesselJ(m,z),z);
```

to compute $J_m'(z)$. Note that the result is expressed in terms of other Bessel functions. How does this formula relate to the formulas given on page 512 of the text? Next, what happens if you compute the second derivative, $J_m''(z)$ using

```
diff(BesselJ(m,z),z$2);
```

How does the result here compare with the relation between $J_m(z)$, $J_m'(z)$, $J_m''(z)$ given by Bessel's equation of order n? (It may help to `simplify` your results.)

More Maple Details

Note that the graph $y = J_m(z)$ for $m \geq 0$ always looks more or less like a decaying oscillation. For us, the important consequence is that as $z \to \infty$, we get a *sequence of positive roots* of the equation $J_m(z) = 0$ (a different sequence for each m). These roots of the Bessel functions will be very important in our study of the vibrating circular drumhead. In addition to the `BesselJ` function, Maple also has a function called `BesselJZeros`. Try entering

```
evalf(BesselJZeros(3,4));
```

The result is a decimal approximation to the 4th positive zero of $J_3(z)$. (Note: "positive" means *strictly positive* here.) Compare with the plot of $y = J_3(z)$ to check this. If you leave out the `evalf` in the command above, Maple will just give you the input back again, since that is an *exact* way to specify the number in question (!) In general `BesselJZeros(m,n);` gives the nth (strictly) positive root of $J_m(z) = 0$.

Lab Questions, part 2

D) (*"The punchline"* – be sure you understand what is happening here and in the following questions E,F carefully.) We introduced the Bessel equation of order m:

$$z^2\frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2)f = 0$$

because we wanted to study the eigenvalues and eigenfunctions for the boundary value problem:

$$(2) \qquad \begin{cases} \frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r} f = -\lambda r f \\ f(a) = 0 \\ |f(0)| < \infty \end{cases} .$$

(We got Bessel's equation by letting $z^2 = \lambda r^2$.) Let

$$L(f) = \frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r} f$$

denote the differential operator in (2). Using the Bessel function zeroes, we can construct the functions

```
f := BesselJ(m,BesselJZeros(m,n)*r/a);
```

Compute $L(f)$ using Maple, and `simplify`. What does the result tell you?

E) Take $m = 4$, $n = 1, 2, 3, 4, 5$ (one at a time of course), $a = 2$ and plot the functions f from part F on the interval $r \in [0, 2]$. Do these functions satisfy the properties we know for eigenfunctions of Sturm-Liouville boundary value problems? Explain.

F) What are the eigenvalues and eigenfunctions of the general problem (2)? Give a complete description by generalizing what you did in parts D and E.

Assignment

Group write-ups for the above lab questions, due April 26.