Mathematics 373 – Applied Mathematics (PDE)
Information on Final Projects
March 13, 2001

General Information

As announced in the course syllabus, one of your assignments for this course will be a group term project, leading to a paper of approximately 10 - 15 pages. You may work individually, or in groups of 2 or 3 on the term project. I will leave the selection of groups up to you, unless problems develop. All members of a group are expected to participate fully in the development of the project, and as a part of the final submission of materials, all members of each group will be asked to give evaluations of the contributions of the others. I will be assigning individual grades for the project based on the paper, and the "peer reviews" from your fellow group members. The following gives a timetable to help you structure your work on this assignment and several suggested topics for you to consider. Other topics of your choice are also possible, but you will need to consult with me *on or before March 22* to make sure the idea you have in mind is acceptable.

Timetable

1) By Thursday, March 22 *at the latest*, notify me of the group you have chosen to work with, and the topic you will be pursuing. I will be happy to make suggestions and/or help you locate materials. Mr. Stankus. the Science Librarian and the other library staff will also be happy to assist you if you need to find other presentations of the material.
2) The week of April 16 (the week after Easter Break), I will schedule a short "progress report" session with each of the groups (at mutually convenient times outside of class). By this time, you should be prepared to discuss what you want to put in the paper. I will be more than happy to meet with you before and after these formal progress report sessions as well to help you tackle your projects.
3) The term project papers will be due Tuesday, May 1. (This is a *firm deadline*. I will not be able to accept late work on this assignment!)

Suggested Topics

1. Numerical Methods for Solving PDE

(This topic will require some extensive Maple calculations to solve big systems of linear equations.) The analytic techniques we have studied in this class for solving PDE (including Fourier series and other eigenfunction expansions) are applicable only in quite limited situations (i.e. for very simple PDE coming from rather unrealistic physical situations). In fact, when applied mathematicians, scientists, or engineers need to solve PDE "in the real world" it is much more common for approximate numerical methods to be used. For this project, you would learn the basic ideas behind the construction of *finite* *difference* numerical methods for the heat equation, the wave equation, Poisson's equation, and so forth. The first step in all of these cases is to *discretize* the space and time variables by subdividing the ranges of these variables (as in the construction of Riemann sum approximations to integrals over the same regions). This introduces a collection of *grid points* where we seek approximate values of the solution function u.

The partial derivatives in the PDE are replaced by corresponding finite difference approximations. For instance if we subdivide in the x and y directions with $\Delta x = \Delta y = h$, then with $x_i = x_0 + ih, y_j = y_0 + jh$, we might use

$$\begin{cases} u_x(x_i, y_j) \doteq \frac{u(x_{i+1},y_j)-u(x_i,y_j)}{h} \\ u_y(x_i, y_j) \doteq \frac{u(x_i,y_{j+1})-u(x_i,y_j)}{h} \end{cases}$$

(called *forward difference* approximations to the first derivatives), or

$$\begin{cases} u_{xx}(x_i, y_j) \doteq \frac{u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)}{h^2} \\ u_{yy}(x_i, y_j) \doteq \frac{u(x_i,y_{j+1})-2u(x_i,y_j)+u(x_i,y_{j-1})}{h^2} \end{cases}$$

(these are called *centered difference* approximations to second derivatives). We substitute these approximations into the PDE and derive a system of equations for the approximate values $u(x_i, y_j)$. If the original PDE was a linear equation, then the corresponding equations are linear algebraic equations, and these can be solved by several different approaches. The details of these methods, as well as the linear algebra techniques that are often used to solve the resulting systems of linear equations is described in Chapter 12 of the text *Numerical Analysis* by Burden and Faires, on reserve for our class in the Science Library. For this project you would work through and present the details of this approach, and solve an applied problem like the one described in Project D in Chapter 10 in our text.

2. Laplace Transform Method for Solving ODE and PDE

(Some portions of this project use ideas from the Complex Analysis course that Prof. Anderson taught last year.) The Laplace transform of a function of t is the function of a new variable s defined by

$$\mathcal{L}[f](s) = F(s) = \int_0^\infty f(t)e^{-st}\, dt$$

The importance of the Laplace transform really comes from its properties, especially

$$\mathcal{L}\left[\frac{df}{dt}\right](s) = sF(s) - f(0)$$

This says the Laplace transform "turns differentiation of f with respect to t into multiplication by of the transform F by s". For ODE's, the Laplace transform gets rid of derivatives entirely – it turns a linear ODE with constant coefficients into an *algebraic equation* that can be solved by elementary means. Then the solution of the ODE can be reconstructed by the "inverse Laplace transform". This is described in Chapter 7 of Nagle, Saff, and Snider, which would be the starting point for this project. For PDE's in two independent variables x, t where one variable is time, the Laplace transform with respect to t takes the PDE to an ODE with respect the the space variable x. This is discussed briefly in Project B. of Chapter 9 of Nagle, Saff, and Snider, and in more detail in Haberman's *Elementary Applied Partial Differential Equations*, which is on reserve in the Science Library for our class. For this project, you would learn the details of this technique, work through proofs of the key properties of the Laplace transform, and present solutions of several PDE by these methods. (Note: The connection with complex analysis comes because it is sometimes convenient to evaluate Laplace transform integrals via complex contour integration and the calculus of residues.)

3. Green's Functions

The solutions to boundary value problems for ODE of the Sturm-Liouville form

$$\begin{cases} \frac{d}{dx}(p(x)\frac{dy}{dx}) + q(x)y = -f(x) \\ c_1 y(a) + c_2 y'(a) = 0 \\ c_3 y(b) + c_4 y'(b) = 0 \end{cases}$$

(and also to many PDE can often be written via integral formulas involving "Green's functions" for the appropriate problem). The Green's function $G(x, x_0)$ itself is the solution to a problem of this form where f is a Dirac delta (unit impulse) "function" at x_0. Given this one solution, we get all solutions via the formula

$$y = \int_a^b G(x,s)f(x)\ ds$$

Green's functions are a very important theoretical and practical tool in more advanced PDE. For this project, you would learn about the basics of this idea, starting with Green's functions for ODE boundary value problems. (See the later part of Chapter 11 in Nagle, Saff, and Snider). Then you would see how the same ideas lead to solutions of PDE such as the heat equation. Chapter 9 in Haberman's *Elementary Applied Partial Differential Equations*, which is on reserve in the Science Library for our class, is a good source.

4. Numerical Methods for ODE Boundary Value Problems

(This topic will require some extensive Maple calculations to implement the shooting method and solve big systems of linear equations.) As mentioned under topic 1 above, the analytic techniques that we have learned in this course are too limited to be applicable in all cases of interest coming from problems in heat conduction, wave motion, etc. This is also the case for ODE boundary value problems. For this project, you would learn two of the basic numerical techniques that people use to derive approximate solutions of problems of this type – the "shooting method" and the finite difference method. The finite difference method is actually very close to the idea described under project 1 above, except that we have only one independent variable. But in the same way, we subdivide the interval where the solution of the BVP is sought, and discretize by replacing the derivatives by finite difference approximations. This leads to a system of algebraic linear equations for the approximate solution values at the grid points. Depending on the boundary conditions, we may have a unique solution that can be found by linear algebra techniques. Or in the Sturm-Liouville situation, the finite differences can lead to an eigenvalue problem. Both of these methods are discussed in Projects D and E in Chapter 11 of our text. They are also covered in first sections of Chapter 11 of *Numerical Analysis* by Burden and Faires, on reserve for our class in the Science Library. For this project, you would write up an exposition of both of these methods, together with several worked examples.

5. Further Steps in the Theory of ODE Boundary Value Problems

(This is the most theoretical of the suggested topics.) In class we talked about the basic points of Sturm-Liouville theory for ODE boundary value problems. For this project, you would take several further steps in the underlying theory. The topics would come from Section 11.8 in our text (oscillation and comparison theory for solutions of Sturm-Liouville problems). The main goal would be to work through and present the proof of the Picone Comparison Theorem (Theorem 17 on page 747) outlined in Project C of Chapter 11.