Mathematics 373 – Applied Mathematics (PDE)
Final Problem Set
April 30, 2001

Groundrules and Directions

This is an open-book, open-notes, open-library, open Swords 335, open Internet, unlimited time assignment. If you use something you find in a book other than our text or on the web in one of your solutions, please footnote it (give the full URL or "web address" for any sites you use). You must work independently, and you may not consult other faculty members in working on these problems. *When you have completed the problem set, please write a brief statement saying that you have abided by these ground rules, sign it, and turn it in with your solutions.*

In all parts of problems where a series solution of a BVP is asked for, give your answer *together with formulas for the coefficients.*

Don't kill trees unnecessarily for the Maple output you submit: Please delete irrelevant work and *resize your plots* before printing.

Due: No later than 5:00 pm, Friday, May 11.

I. (35) In this problem, we will reconsider solutions of the 1D wave equation on finite intervals and compare the information we get from Fourier series solutions with the information we get from D'Alembert's approach to the wave equation on $\mathbf{R}$.

A) Consider the problem

$$(1)\qquad \begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(5,t) = 0 \\ u(x,0) = \begin{cases} 5(x-2)(3-x) & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \\ u_t(x,0) = 0 \end{cases}$$

on the interval $[0,5]$. Using Maple, compute and plot a Fourier (sine) series solution to this problem. (Be sure to take enough terms in your partial sum so that there is good agreement with the initial conditions.) Describe the behavior of the solution as t ranges from 0 to 10. What happens when a wave reaches one of the endpoints of the string?

Now we want to study the relation between solutions of a general problem of the same type as (1):

$$(2)\qquad \begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(L,t) = 0 \quad \text{all } t \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

and the D'Alembert solution of the general initial value problem problem for the 1D wave equation on $\mathbf{R}$

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = F(x) \\ u_t(x,0) = G(x). \end{cases} \tag{3}$$

Recall, D'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}(F(x-t) + F(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} G(s)\ ds$$

(we have taken the speed of wave propagation, c, to be $= 1$ in the general formula). D'Alembert's formula describes a solution on the whole real line; it *does not* take into account the fixed ends of the string, say at $x = 0, x = L$. To bring those additional conditions into play, we will proceed as follows. First, we extend f, g from (2) to *odd functions* on the interval $[-L, L]$ as in the construction of the Fourier sine series. Then we extend the odd extensions to *odd periodic* functions on $\mathbf{R}$ with period $2L$.

B) Sketch the odd periodic extension of the initial data in the given problem (1) above.

Apply the D'Alembert formula with F, G equal to the odd periodic extensions of the initial data from (2) and get the solution of (3) on $\mathbf{R}$ for the periodic initial conditions.

C) Show that the *restriction of the D'Alembert solution* of (3) to the interval $[0, L]$, with initial data given by the odd periodic extensions F, G is the corresponding solution of (2).

D) Show how part C leads to the form of the solution you saw in part A for the specific problem (1).

II. (30) Consider a Sturm-Liouville form differential equation:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0, \tag{4}$$

and assume $p(x) > 0$ for all $x \in [a, b]$ (as in a regular Sturm-Liouville eigenvalue problem). In this problem, we will study the *zeroes* of solutions of (4), and prove a result called the *Sturm Separation Theorem* – part C below. Let y_1, y_2 be any two solutions of (4). As we saw in the ODE class, the *Wronskian* of y_1, y_2 is defined to be:

$$W[y_1, y_2](x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ \frac{dy_1}{dx}(x) & \frac{dy_2}{dx}(x) \end{pmatrix}$$

A) Show that $W[y_1, y_2](x) = C/p(x)$ for some constant C. (**Hint:** What is the derivative $\frac{d}{dx}W[y_1, y_2](x)$?)

B) Use part A to show that if the Wronskian of y_1 and y_2 is zero at any $c \in [a, b]$, then y_1 and y_2 are *linearly dependent.* You will need to use the fact that for all $c \in [a, b]$, and all $A, B \in \mathbf{R}$, there is a *unique* solution of (4), together with the initial conditions $y(c) = A$, $\frac{dy}{dx}(c) = B$.

C) Let y_1, y_2 be two linearly independent solutions of (4). The Sturm Separation Theorem is the statement that between any two consecutive zeroes of y_1 in $[a, b]$, there is precisely one zero of y_2. Argue as follows: Assume $x_1 < x_2$ are the consecutive zeroes of y_1, but that y_2 is not zero on the interval $[x_1, x_2]$. Consider the function $\frac{y_1}{y_2}$ and apply the "most valuable theorem" (MVT) of calculus.

III. (35) For the 2D heat equation on a circular disk of radius a, if the outer edge is *insulated*, we are led to consider the problem

$$\begin{cases} u_t = k\nabla^2 u \\ u_r(a, \theta, t) = 0 \\ u(r, \theta, 0) = \alpha(r, \theta) \end{cases}$$

A) If we look for product solutions $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, what ODE's must R, Θ, and T satisfy, and what boundary value problem for $R(r)$ results?

B) Show that for each Bessel function of the first kind $J_m(z)$ there is an infinite sequence of numbers $\mu_{mn} \geq 0$ where $J'_m(\mu_{mn}) = 0$. (Note: I'm looking for an actual proof here, not just a plausibility argument from the shape of the graph of $J_m(z)$.)

C) What are the eigenvalues and eigenfunctions of the 1D boundary value problem for $R(r)$?

D) Give a Fourier-Bessel expansion for the solution of the problem:

$$\begin{cases} u_t = k\nabla^2 u \\ u_r(a, \theta, t) = 0 \\ u(r, \theta, 0) = e^{-(r-1/2)^2}. \end{cases}$$

Taking $a = k = 1$, plot a partial sum approximation to the solution using Maple. Show the temperature function at the times $t = 0, 0.5, 1.0, 1.5$, and 2.0. (Or do a Maple animation!)

Extra Credit. (10) Who first posed the question, "can you hear the shape of a drum?" When, and by whom, was the question answered? What is the answer? What areas of mathematics were involved in finding the answer?

Have an enjoyable and productive summer!