Mathematics 373 – Applied Mathematics (PDE)
Discussion 2 – Separation of Variables and Boundary Value Problems
February 8, 2001

Background

Last time in class, we introduced the following "program" for finding solutions of initial-boundary value problems for homogeneous linear PDE such as the heat and wave equations:

1) First, we look for "completely separated" solutions of the relevant PDE: $u(x,t) = X(x)T(t)$
2) Among the separated solutions, we look for those satisfying the boundary conditions,
3) Then, to derive solutions satisfying given initial conditions we try to form linear combinations of the solutions found in Step 2.

For example, for the problem:

$$(1) \qquad \begin{cases} u_t = ku_{xx} & \text{(heat equation)} \\ u(x,0) = f(x) & \text{(given initial temperature)} \\ u(0,t) = u(L,t) = 0 & \text{all } t \text{ (ends of wire held at zero temperature)} \end{cases}$$

we saw that $u(x,t) = X(x)T(t)$ solves $u_t = ku_{xx}$ if and only if

$$\begin{cases} T'(t) = -\lambda k T(t) \\ X''(x) = -\lambda X(x). \end{cases}$$

This says

$$\begin{cases} T(t) = e^{-\lambda k t} & \text{(up to a constant multiple)} \\ X(x) = \begin{cases} Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0 \\ A + Bx & \text{if } \lambda = 0 \\ A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) & \text{if } \lambda > 0 \end{cases} \end{cases}$$

But the boundary conditions $u(0,t) = u(L,t) = 0$ imply $X(0) = X(L) = 0$, and this happens with $X(x)$ not identically zero *only in the case that*

$$\sin(\sqrt{\lambda}L) = 0 \Leftrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

for some $n \geq 1$. We get the corresponding separated solutions of the heat equation and the boundary conditions:

$$u_n(x,t) = e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

If coefficients B_n can be found to make

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{2}$$

(the initial temperature), then

$$u(x,t) = \sum_{n=1}^{\infty} B_n u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right) \tag{3}$$

is the solution of (1) that we are looking for! (Substitute $t = 0$ and compare with (2).)

Some terminology:

a) The values $\lambda = \left(\frac{n\pi}{L}\right)^2$ are called the *eigenvalues* of the 1D BVP $X''(x) = -\lambda X(x)$ with $X(0) = X(L) = 0$. *$X(x)$ satisfies the boundary conditions only when λ is one of these eigenvalues.*

b) The corresponding functions $X(x) = \sin(\sqrt{\lambda}x) = \sin\left(\frac{n\pi x}{L}\right)$ are called the *eigenfunctions* of the BVP.

Today, we want to work through several other examples of this separation of variables technique and find eigenvalues and eigenfunctions of the BVP's we get.

Discussion Questions

A) In some simple cases, we can find appropriate B_n in (2) more or less "by inspection". For instance, what is the solution of the problem (1) if $L = \pi$, and

$$f(x) = 3\sin(2x) - 4\sin(5x)?$$

Find any one set of coefficients B_n that "works"; we'll ask whether there is more than one such set of coefficients later.

B) Next, consider the following problem

$$\begin{cases} u_t = k u_{xx} & \text{(heat equation)} \\ u(x,0) = f(x) & \text{(given initial temperature)} \\ u_x(0,t) = u_x(L,t) = 0 & \text{all } t \text{ (ends of wire \textit{insulated})} \end{cases} \tag{4}$$

What are the eigenvalues and eigenfunctions for (4)? (You'll need to follow what we did for the boundary conditions $u(0,t) = u(L,t) = 0$; but there will be differences.) What does the analog of equation (3) look like here – the linear combination giving a solution of (4).

C) The same techniques also apply to BVP's for the wave equation. Consider

$$(6)\quad \begin{cases} u_{tt} = ku_{xx} & \text{(wave equation)} \\ u(x,0) = f(x) & \text{(given initial displacement)} \\ u_t(x,0) = g(x) & \text{(given initial velocity)} \\ u(0,t) = u(L,t) = 0 & \text{all } t \text{ (ends of string fixed at zero displacement)} \end{cases}$$

What are the 2 ODE's we get by separating variables here? What are the eigenvalues and eigenfunctions for the equation for $X(x)$, with $X(0) = X(L) = 0$? Show that the corresponding solutions of the wave equation and the boundary conditions can be written as

$$(7)\qquad u_n(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

To get a (possibly infinite) linear combination of these $u_n(x,t)$ to satisfy the initial conditions, A_n and B_n must satisfy equations like (2) above. What do these look like here?

D) Note that the solutions (7) of the wave equation represent strings that keep oscillating in an exactly periodic fashion forever. In the real world, that never happens because of additional forces that we ignored in the derivation of the wave equation – e.g. air resistance, frictional forces at the ends, etc. In this problem, we will study a PDE modeling a string moving in the presence of damping force(s) proportional to the string's velocity:

$$u_{tt} = c^2 u_{xx} - \beta u_t$$

where $\beta > 0$. We want to solve boundary value problems of the form

$$\begin{cases} u_{tt} = c^2 u_{xx} - \beta u_t \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

1) Show that if we apply the separation of variables method, looking for separated solutions $u(x,t) = X(x)T(t)$, then we get the ordinary differential equations

$$T''(t) + \beta T'(t) + \lambda c^2 T(t) = 0$$

and

$$X''(x) = -\lambda X(x)$$

2) Show that in order to satisfy the boundary conditions at $x = 0, x = L$, $\lambda = \left(\frac{n\pi}{L}\right)^2$ and $X(x) = \sin\left(\frac{n\pi x}{L}\right)$ for some $n \geq 1$.
3) Suppose that β is relatively small –

$$\beta < \frac{2\pi c}{L}.$$

What do the solutions of the equation for T look like in this case?

Assignment

Group writeups due Tuesday, February 20.