Mathematics 373 – Applied Mathematics (PDE)
Discussion 1 – Diffusion on $\mathbf{R}$
February 1, 2001

Background

Before we continue on to the study of boundary value problems for the wave equation in 1 space variable, we want to introduce the other major PDE that we will focus on this term. This other equation is often called the *heat equation* because it serves as a model for *heat conduction* and was originally introduced in that context. To derive the equation, though, we will consider a different but related situation where our physical intuition may be somewhat more immediately applicable.

Namely, let's consider a motionless liquid filling a long, straight, thin tube with constant cross-sections (their exact shape is not relevant, but you can think circles if that is helpful). Imagine that a chemical substance (e.g. a dye) is introduced at some time and begins to spread out, or *diffuse* through the liquid.[1] To describe the diffusion process, we consider the *concentration* of the dye. *Note:* the hypothesis that the tube is *thin* means that we will ignore possible variation of the concentration on each cross-section of the tube perpendicular to the x axis. We assume that concentration is constant on each cross section, so it depends on only one space variable x – the longitudinal distance along the tube, and on time. This means that the concentration is a function $u(x,t)$, and the units of concentration are *mass per unit length* (volume of a section of the tube is proportional to its length). The *key fact* about diffusion is that:

The dye will tend to spread out from areas of high concentration to areas of low concentration

The precise version of this is an empirical relation (i.e. experimental observation) called *Fick's Law*:

(FL) *The amount of dye traveling to the right past $x = x_0$ per unit time at time t_0 is proportional to the derivative $u_x(x_0, t_0)$.* Note that u_x is the "gradient" of u with respect to the spatial variable in this 1D setting. Dye traveling to the *left* corresponds to a negative amount flowing to the right.

Today, we want to use this information derive a PDE governing u, and consider some of its solutions.

Discussion Questions

A) What does the function $u(x,0)$ represent in the diffusion set-up? Same question for $u(0,t)$. What does $u_t(x,t)$ represent? What does $u_x(x,t)$ represent? If we write Fick's

[1] To be precise, the physical mechanism for the spreading out is the essentially random motions of the individual molecules of the liquid "pushing the dye along"; the liquid as a whole is *not moving* – there are no currents in the tube itself.

Law as an equation, what is the sign of the proportionality constant, recalling that dye always flows from areas of higher concentration to areas of lower concentration?

B) Consider the section of the tube extending from x to $x_1 = x + \Delta x$. Show that the *total mass* of the dye *in that section* (which depends on t), is computed by the integral

$$M(t) = \int_x^{x_1} u(x,t)\ dx \tag{1}$$

(Argue as in the derivation of integral formulas for areas, volumes, etc.: Subdivide, approximate, recognize your approximation as a Riemann sum for an integral, then "let $n \to \infty$" to get the exact mass.)

C) From B, it follows that the rate of change of M with respect to t is given by

$$M'(t) = \frac{d}{dt}\int_x^{x_1} u(x,t)\ dx = \int_x^{x_1} u_t(x,t)\ dx \tag{2}$$

(That is, to differentiate M with respect to t, we can integrate the time derivative of the concentration.) We will assume that no dye enters or leaves our section of the tube through its lateral surface. Thus dye enters our section only by flowing in across the ends at x and x_1. Using this and (FL) (Fick's Law) above, argue that

$$M'(t) = \text{flow in} - \text{flow out} = ku_x(x_1,t) - ku_x(x,t)$$

(Be careful: This equation *does not say that the flow in is $ku_x(x_1,t)$, just that the difference of the flow in and the flow out is the difference on the right.* What does $u_x > 0$ at a point mean? What does $u_x < 0$ mean? What sign does the proportionality factor k have here?)

D) From (1) and (2),

$$\int_x^{x_1} u_t(x,t)\ dx = ku_x(x_1,t) - ku_x(x,t)$$

Divide both sides by $x_1 - x$ and let $x_1 \to x$ to derive the *diffusion equation*:

$$u_t = ku_{xx} \tag{3}$$

CAUTION: This equation is also a 2nd order linear PDE, but it involves just the *first* derivative of u with respect to t, NOT u_{tt} as in the wave equation. This makes a large difference! The techniques for solving the wave equation that we have studied *do not apply* to this equation.

E) For today, we will consider solutions $u(x,t)$ of (3) for all $x \in \mathbf{R}$, but just $t > 0$. Show that each of the following functions is a *solution* of the diffusion equation (3). Also answer each of the questions connected with the specific solutions.

1) $u_1(x,t) = ax + b$ (a,b constant). What is "special" about this type of solution? (What happens to u as t changes?)
2) $u_2(x,t) = x(x^2 + 6kt)$. Is this a physically reasonable concentration function for a diffusion situation? If so, what situation does it describe?
3) $u_3(x,t) = e^{-kt}\sin(x) + 2$ and $u_4(x,t) = e^{-kt}\cos(x) + 2$.
4)
$$u_5(x,t) = \frac{1}{\sqrt{t}} e^{\frac{-x^2}{4kt}}$$
Note that u_5 is not defined at $t = 0$. Plot $u_5(x,t)$ for $k = 1$ and several different t (at least $t = 1/4, 1/2, 1, 2$) and describe what is happening as t increases. What do you think the graph of $\lim_{t\to 0^+} u_5(x,t)$ looks like?

Assignment

Group writeups due in class Thursday, February 8.