College of the Holy Cross, Spring 2018
Math 134 Midterm Exam 3 – Solutions
Friday, April 13

I. Compute each of the following integrals using the trigonometric reduction formulas (on separate sheet) and other methods as appropriate.

A. (15) $\int \sin^4(5x)\ dx$

Solution: First let $u = 5x$ and then $dx = (1/5)\ du$. Then using the SC1 reduction formula, the integral is

$$\frac{1}{5}\left(\frac{-\sin^3(u)\cos(u)}{4} + \frac{3}{4}\left(\frac{-\sin(u)\cos(u)}{2} + \frac{u}{2}\right)\right) + C.$$

Substituting $u = 5x$ back in and simplifying the final answer is

$$\frac{-\sin^3(5x)\cos(5x)}{20} + \frac{-3\sin(5x)\cos(5x)}{40} + \frac{3x}{8} + C$$

B. (15) $\int x\sec^4(3x^2)\ dx$

Solution: First let $u = 3x^2$, so $x\ dx = \frac{1}{6}\ du$. Then using the ST3 reduction formula, the integral is

$$\frac{1}{6}\int \sec^4(u)\ du = \frac{\tan(u)\sec^2(u)}{18} + \frac{\tan(u)}{9} + C$$

This is

$$= \frac{\tan(3x^2)\sec^2(3x^2)}{18} + \frac{1}{9}\tan(3x^2) + C.$$

II. (Trigonometric Substitutions)

A. (10) What trigonometric substitution would you use to evaluate

$$\int \frac{x^2}{\sqrt{64+x^2}}\ dx?$$

Make the substitution and simplify so there is no square root in the resulting trigonometric integral.

Solution: The form of $\sqrt{64+x^2}$ indicates that we want $x = 8\tan\theta$. Then $dx = 8\sec^2\theta\ d\theta$, and the integral becomes

$$\int \frac{(8\tan\theta)^2}{\sqrt{64+64\tan^2\theta}}\ 8\sec^2\theta\ d\theta = 64\int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\ d\theta$$

This could also be simplified to

$$64\int \tan^2\theta\sec\theta\ d\theta.$$

B. (10) Suppose you had used the trig substitution $x = 3\sin\theta$ and you integrated the resulting trigonometric integral to get

$$\ln|\csc\theta - \cot\theta| + C$$

What is the integral expressed as a function of x?

Solution: The reference triangle has x on the opposite side, 3 on the hypotenuse, and $\sqrt{9-x^2}$ on the adjacent side. Hence $\csc\theta$ equals the hypotenuse over the opposite side and $\cot\theta$ equals the adjacent side over the opposite side. The function is

$$\ln\left|\frac{3}{x} - \frac{\sqrt{9-x^2}}{x}\right| + C.$$

III. (Partial Fractions)

A. (5) To integrate $\dfrac{x^4+1}{x^3+7x+1}$, what would be the first step? Carry the step out and give the rational function to which you would apply the partial fraction decomposition.

Solution: Since the degree of the top is 4 and the degree of the bottom is 3, we need to divide x^3+7x+1 into x^4+1. The quotient is x and the remainder is $-7x^2-x+1$. The result of the division is:

$$\frac{x^4+1}{x^3+7x+1} = x + \frac{-7x^2-x+1}{x^3+7x+1}.$$

B. (10) To decompose $\dfrac{1}{(x+4)^2(x+1)^3(x^2+9)}$ into partial fractions, what would the form of the fractions be (leave coefficients as undetermined; *do not try to solve for them*).

Solution: From the form of the factorization,

$$\frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3} + \frac{Fx+G}{x^2+9}.$$

C. (10) Determine the values A, B, C making

$$\frac{3x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Solution: Clearing denominators, we have

$$\begin{aligned} 3x+1 &= A(x^2+4) + (Bx+C)x \\ &= (A+B)x^2 + Cx + 4A. \end{aligned}$$

Equating coefficients of like powers of x gives:

$$A+B=0, C=3, 4A=1.$$

The second and third equations give $A = 1/4$ and $C = 3$. Then from the first equation $B = -1/4$.

D. (15) You have decomposed a rational function $f(x)$ into partial fractions as

$$f(x) = x^2 + 3x + \frac{1}{(x+3)^2} + \frac{2}{(x+3)^3} + \frac{4x+3}{x^2+1}.$$

What is $\int f(x)\,dx$?

Solution: By the power rule (u-substitution) and the inverse tangent form,

$$\int f(x)\,dx = \frac{x^3}{3} + \frac{3x^2}{2} - \frac{1}{x+3} - \frac{1}{(x+3)^2} + 2\ln|x^2+1| + 3\tan^{-1}(x) + C$$

IV. (Improper Integrals)

A. (5) Is there a finite area between the x-axis and the graph $y = \frac{1}{x^{2/3}}$ for $0 < x \leq 1$? Set up the appropriate (improper) integral and determine if it converges.

Solution: The necessary integral is

$$\int_0^1 \frac{1}{x^{2/3}}\,dx = \int_0^1 x^{-2/3}\,dx$$

This is improper because of the discontinuity at the lower limit of integration. So we must determine whether

$$\lim_{a\to 0^+} \int_a^1 x^{-2/3}\,dx$$

exists. Computing the integral we have

$$\lim_{a\to 0^+} 3x^{1/3}\Big|_a^1 = \lim_{a\to 0^+} 3 - 3a^{1/3} = 3$$

Since the limit exists, the integral converges. The area is finite and equals 3.

B. (5) Does the solid of revolution obtained by rotating the region from part A about the x-axis have a finite volume or not? Why?

Solution: The volume would be computed by

$$\int_0^1 \pi(x^{-2/3})^2\,dx = \pi\int_0^1 x^{-4/3}\,dx.$$

This is improper for the same reason as the integral in part A and we determine whether it converges by a similar method. But

$$\int_0^1 x^{-4/3}\,dx = \lim_{a\to 0^+} -3x^{-1/3}\Big|_a^1 = \lim_{a\to 0^+} -3 + 3a^{-1/3},$$

which does not exist because the $a^{-1/3} = \frac{1}{a^{1/3}}$ is now in the denominator. This integral *diverges*(!) In this case, we get a finite area such that the corresponding solid of revolution does not have finite volume.