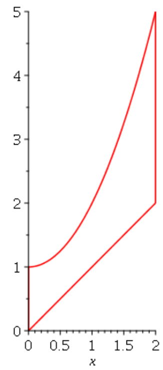


Figure 1: The region for problem I.

**College of the Holy Cross, Spring 2018**
**Math 134 – Solutions for Midterm Exam 2**
**Friday, March 16**

I. All parts of this problem refer to the region $R$ bounded by $y = x$, $y = x^2 + 1$, $x = 0$ and $x = 2$.

A. (10) Sketch the region $R$.

*Solution:* See Figure 1.

B. (10) Compute the area of the region $R$.

*Solution:* The area is computed by

$$\begin{aligned}\text{Area} &= \int_0^2 x^2 + 1 - x\, dx \\ &= \frac{x^3}{3} + x - \frac{x^2}{2}\Big|_0^2 \\ &= \frac{8}{3} + 2 - 2 - 0 \\ &= \frac{8}{3}.\end{aligned}$$

C. (10) Compute the volume of the solid of revolution obtained by rotating the region $R$ about the $x$-axis.

*Solution:* The cross-sections by planes $x =$ constant are washers with inner radius $x$

and outer radius $x^2 + 1$, so the volume is

$$\begin{aligned}
\text{Volume} &= \int_0^2 \pi(x^2+1)^2 - \pi x^2 \, dx \\
&= \pi \int_0^2 x^4 + x^2 + 1 \, dx \\
&= \pi \left( \frac{x^5}{5} + \frac{x^3}{3} + x \right|_0^2 \\
&= \pi \left( \frac{32}{5} + \frac{8}{3} + 2 - 0 \right) \\
&= \frac{(96 + 40 + 30)\pi}{15} \\
&= \frac{166\pi}{15}.
\end{aligned}$$

D. (10) Compute the volume of the solid of revolution obtained by rotating the region $R$ about the line $x = -2$. (Note: Any correct method is OK here.)

*Solution:* The easier way to do this is to use cylindrical shells:

$$\begin{aligned}
\text{Volume} &= \int_0^2 2\pi(x+2)(x^2+1-x) \, dx \\
&= 2\pi \int_0^2 x^3 + x^2 - x + 2 \, dx \\
&= 2\pi \left( \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + 2x \right|_0^2 \\
&= 2\pi \left( 4 + \frac{8}{3} - 2 + 4 - 0 \right) \\
&= \frac{52\pi}{3}.
\end{aligned}$$

This can also be done by slices, but the inner or outer radii of the washer cross-sections *change* at $y = 1$, $y = 2$, so the volume would be computed as the sum of three integrals like this:

$$\begin{aligned}
\text{Volume} = &\int_{y=0}^{y=1} \pi(2+y)^2 - \pi(2)^2 \, dy + \int_{y=1}^{y=2} \pi(2+y)^2 - \pi(2+\sqrt{y-1})^2 \, dy \\
&+ \int_{y=2}^{y=5} \pi(2+2)^2 - \pi(2+\sqrt{y-1})^2 \, dy.
\end{aligned}$$

It's a good exercise to verify that this gives the same value as the shell method, but there's a lot of algebra involved(!)

II. Compute each of the following integrals by an appropriate method (e.g. $u$-substitution, parts, etc.)

A. (15) $\int x^2 \cos(5x^3)\ dx$

*Solution:* This one is one where you just want a $u$-substitution. Let $u = 5x^3$, then $du = 15x^2\ dx$ and the integral becomes

$$\int \cos(u)\frac{1}{15}\ du = \frac{1}{15}\sin(u) + C = \frac{1}{15}\sin(5x^3) + C.$$

B. (15) $\int x^2 \cos(5x)\ dx$

*Solution:* For this one, we use parts (twice), letting $u = x^2$ the first time and $dv = \cos(5x)\ dx$. Then $du = 2x\ dx$ and $v = \frac{1}{5}\sin(5x)$. So

$$\int x^2 \cos(5x)\ dx = \frac{x^2 \sin(5x)}{5} - \frac{2}{5}\int x \sin(5x)\ dx.$$

Now use parts again with $u = x$ and $dv = \sin(5x)$; the final answer is

$$\int x^2 \cos(5x)\ dx = \frac{x^2 \sin(5x)}{5} + \frac{2x\cos(5x)}{25} - \frac{2}{125}\sin(5x) + C.$$

C. (15) $\int e^{\sqrt{x}}\ dx$ (Hint: Let $u = x^{1/2}$.)

*Solution:* If we do the indicated substitution first we get $du = \frac{1}{2}x^{-1/2}\ dx$ so $dx = 2x^{1/2}du = 2udu$. Rewriting the given integral in terms of $u$, we get $\int 2ue^u\ du$, which is a simple parts form:

$$2\int ue^u\ du = 2ue^u - 2e^u + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

III.

A. (10) Identify a $u$ and a $dv$, then carry out an integration by parts to derive the following reduction formula for exponents $k \geq 1$.

$$\int (\ln(x))^k\ dx = x(\ln(x))^k - k\int (\ln(x))^{k-1}\ dx$$

*Solution:* The appropriate choice is $u = (\ln(x))^k$ and $dv = dx$. Then $du = k(\ln(x))^{k-1}\frac{1}{x}\ dx$ (chain rule), and $v = x$. Applying the parts formula,

$$\int (\ln(x))^k\ dx = x(\ln(x))^k - k\int (\ln(x))^{k-1}\frac{1}{x}x\ dx$$

The $\frac{1}{x}$ and the $x$ cancel leaving the given form.

B. (5) Apply the formula in part A to integrate $\int (\ln(x))^2\ dx$. (Note: You can do this part even if you didn't see how to finish part A.)

*Solution:* We apply the formula first with $k = 2$, then again with $k = 1$:

$$\begin{aligned}\int (\ln(x))^2\ dx &= x(\ln(x))^2 - 2\int \ln(x)\ dx \\ &= x(\ln(x))^2 - 2\left(x\ln(x) - \int\ dx\right) \\ &= x(\ln(x))^2 - 2x\ln(x) + 2x + C.\end{aligned}$$