Mathematics 136 – Calculus 2
Solutions for Practice Problems from Exam 1 Review Sheet
February 14, 2020

I.

A) Since the interval is $[0, 1]$ and $n = 4$, the sums are

Left endpoints:

$$f(0)(.25) + f(.25)(.25) + f(.5)(.25) + f(.75)(.25) = -.53125$$

Right endpoints:

$$f(.25)(.25) + f(.5)(.25) + f(.75)(.25) + f(1)(.25) = -.78125$$

Midpoints:

$$f(.125)(.25) + f(.375)(.25) + f(.625)(.25) + f(.875)(.25) = -.671875$$

B) $f'(x) = 2x - 2 \leq 0$ for all $x \in [0, 1]$. Hence f is *decreasing* on this interval. This implies that the left-hand Riemann sum is greater than $\int_0^1 x^2 - 2x\ dx$ (i.e. less negative than the integral), and the right-hand Riemann sum is less (i.e. more negative) than the value of the integral. Using the Evaluation Theorem, we can check this:

$$\int_0^1 x^2 - 2x\ dx = \frac{x^3}{3} - x^2 \Big|_0^1 = -\frac{2}{3} \doteq -.66667$$

The left-hand sum is greater (less negative) and the right-hand sum is less (more negative) than this value.

C) The sum is a right-hand Riemann sum for the function $f(x) = \frac{\cos(x)}{x^2+1}$ on the interval $[0, \pi]$. So it will equal

$$\int_0^\pi \frac{\cos(x)}{x^2+1}\ dx.$$

II.

A) We have $\Delta t = 2$ and

$$L_5 = v(0) \cdot 2 + v(2) \cdot 2 + v(4) \cdot 2 + v(6) \cdot 2 + v(8) \cdot 2 \doteq 29.62$$

(meters).

B) The exact value is

$$\int_0^1 0\sqrt{t+5}\ dt = \frac{2}{3}(t+5)^{3/2}\Big|_0^{10} = \frac{2}{3}(15\sqrt{15} - 5\sqrt{5}) \doteq 31.28$$

III.

A) We have

$$\sum_{i=1}^{N}\left(\frac{5i}{N}\right)^2\cdot\frac{5}{N} = \frac{125}{N^3}\sum_{i=1}^{N} i^2$$
$$= \frac{125}{N^3}\cdot\frac{N(N+1)(2N+1)}{6}$$
$$= \frac{125}{3}+\frac{125}{2N}+\frac{125}{6N^2}.$$

Hence the limit as $N\to\infty$ is $\frac{125}{3}$.

B) The limit in part A is the limit of a R_N Riemann sum and computes the integral:

$$\int_0^5 x^2\,dx = \left.\frac{x^3}{3}\right|_0^5 = \frac{125}{3}.$$

IV. Let

$$f(x) = \begin{cases} 1 & \text{if } 0\le x\le 3 \\ x-2 & \text{if } 3\le x\le 5 \\ 13-2x & \text{if } 5\le x\le 8 \end{cases}$$

(A) Sketch the graph $y = f(x)$. (Omitted – the graph is made up of segments of three different straight lines.)
In the rest of the parts, $F(x) = \int_0^x f(t)\,dt$, where f is the function from part A.

(B) Assuming $F(0) = 0$, Compute $F(1), F(2), F(3), F(4), F(5), F(6), F(7), F(8)$ given the information in the graph of f.

Using the area interpretation of the definite integral we have

$$F(1) = \int_0^1 f(x)\,dx = 1$$

$$F(2) = \int_0^2 f(x)\,dx = 2$$

$$F(3) = \int_0^3 f(x)\,dx = 3$$

$$F(4) = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx = 3+\frac{3}{2} = \frac{9}{2}$$

$$F(5) = \int_0^4 f(x)\,dx + \int_4^5 f(x)\,dx = \frac{9}{2}+\frac{5}{2} = 7$$

$$F(6) = \int_0^5 f(x)\,dx + \int_5^6 f(x)\,dx = 7+2 = 9$$

$$F(7) = \int_0^6 f(x)\,dx + \int_6^{13/2} f(x)\,dx + \int_{13/2}^7 f(x)\;dx = 9 + \frac{1}{4} - \frac{1}{4} = 9$$

$F(8) = \int_0^5 f(x)\,dx + \int_5^{13/2} f(x)\,dx + \int_{13/2}^8 f(x)\,dx = \int_0^5 f(x)\,dx = 7$ (the last two integrals cancel since they represent equal areas with opposite signs).

(C) Are there any critical points of F? If so, find them and say whether they are local maxima, local minima, or neither. If not, say why not.

By the Fundamental Theorem of Calculus, $F'(x) = f(x)$. Since $f(13/2) = 0$, the point $x = 13/2$ is a critical point. Since $F' = f$ changes sign from positive to negative at the critical point, $x = 13/2$ is a local maximum.

(D) How is the graph of $F(x)$ related to the graph of

$$G(x) = \int_2^x f(t)\;dt?$$

This is another antiderivative of f. It satisfies $F(x) = \int_0^2 f(t)\;dt + G(x)$ by the interval union property for integrals. So the graph of $F(x)$ would be the graph of $G(x)$ shifted up by $\int_0^2 f(t)\;dt = F(2) = 2$.

V. Find the derivatives of the following functions.

(A) $f(x) = \int_0^x \sin(t)/t\,dt$.

$$f'(x) = \frac{\sin x}{x}$$

(B) $g(x) = \int_5^{x^3} \tan^4(t)\;dt$.

$g(x) = m(x^3)$, where $m(x) = \displaystyle\int_5^x \tan^4(t)\;dt$. Then, $g'(x) = m'(x^3)\cdot 3x^2 = \tan^4(x^3)\cdot 3x^2$.

(C)) $h(x) = \int_{-3x}^{5x} e^{t^2}\sin(t)\;dt$.

$h(x) = n(x) + l(x)$, where $n(x) = \displaystyle\int_{-3x}^0 e^{t^2}\sin(t)\;dt$ and $l(x) = \displaystyle\int_0^{5x} e^{t^2}\sin(t)\;dt$. Then $h'(x) = n'(x) + l'(x) = -(e^{(-3x)^2}\sin(-3x))\cdot(-3) + 5\cdot e^{(5x)^2}\sin(5x) = 3e^{9x^2}\sin(-3x) + 5e^{25x^2}\sin(5x)$.

VI.

(A) Compute $\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\;dx$

$$\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\;dx = x^5 - 2x^{3/2} + e^x + 2\ln|x| + C$$

(B) Apply a u-substitution to compute $\int x(4x^2-3)^{3/5}\;dx$

$u = 4x^2 - 3,\ du = 8x\,dx$. Then $\displaystyle\int x(4x^2-3)^{3/5}\ dx = \int \frac{1}{8}u^{3/5}\,dx = \frac{1}{8}\frac{u^{8/5}}{8/5} + C = \frac{5}{64}(4x^2-3)^{8/5} + C$

(C) Integrate by parts to compute $\int x^2 e^{6x}\ dx$. We let $u = x^2$ and $dv = e^{6x}$ the first time, then integrate by parts again with $u = x$, $dv = e^{6x}$. The result is

$$\begin{aligned}\int x^2 e^{6x}\ dx &= \frac{x^2e^{6x}}{6} - \frac{1}{3}\int xe^{6x}\ dx \\ &= \frac{x^2e^{6x}}{6} - \frac{xe^{6x}}{18} + \frac{1}{18}\int e^{6x}\ dx \\ &= \frac{x^2e^{6x}}{6} - \frac{xe^{6x}}{18} + \frac{e^{6x}}{108} + C\end{aligned}$$

(D) Integrate with an appropriate u-substitution:

$$\int \frac{t^2+1}{t^3+3t+3}\ dt?$$

Let $u = t^3+3t+3,\ du = (3t^2+3)dt = 3(t^2+1)dt$. Then $\displaystyle\int \frac{t^2+1}{t^3+3t+3}\ dt = \frac{1}{3}\int \frac{1}{u}\ du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|t^3-3t+3| + C.$

(E) Compute $\int x^4 \ln(3x)\ dx$ using integration by parts.

Let $u = \ln(3x)$, so $du = \frac{1}{x}\ dx$. Then $dv = x^4\ dx$, and $v = \frac{x^5}{5}$. Then

$$\int x^4\ln(3x)\ dx = \frac{x^5\ln(3x)}{5} - \int \frac{x^4}{5}\ dx = \frac{x^5\ln(3x)}{5} - \frac{x^5}{25} + C.$$

(F) Using the substitution $u = 3x$ and the SC1 reduction formula twice:

$$\begin{aligned}\int \sin^4(3x)\ dx &= \frac{1}{3}\int \sin^4(u)\ du \\ &= \frac{1}{3}\left(\frac{-\sin^3(u)\cos(u)}{4} + \frac{3}{4}\int \sin^2(u)\ du\right) \\ &= \frac{1}{3}\left(\frac{-\sin^3(u)\cos(u)}{4} + \frac{3}{4}\left(\frac{-\sin(u)\cos(u)}{2} + \frac{u}{2}\right)\right) \\ &= -\frac{\sin^3(u)\cos(u)}{12} - \frac{\sin(u)\cos(u)}{8} + \frac{u}{8} + C \\ &= -\frac{\sin^3(3x)\cos(3x)}{12} - \frac{\sin(3x)\cos(3x)}{8} + \frac{3x}{8} + C\end{aligned}$$

V. Compute each of the integrals below using some combination of basic rules, substitution, integration by parts, the table of integrals, partial fractions, and trigonometric substition. You must show all work for full credit.

A)

$$\int \frac{e^{\sqrt{\sin(x)}}\cos(x)}{\sqrt{\sin(x)}}\,dx$$

Use the substitution $u = \sqrt{\sin x}$, $du = \dfrac{1}{2\sqrt{\sin x}}\cos x\,dx$. Then

$$\int \frac{e^{\sqrt{\sin(x)}}\cos(x)}{\sqrt{\sin(x)}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{\sin x}} + C.$$

B) Use a substitution $u = 2x$, then either SC3 or SC4. With SC4, $m = n = 2$, then SC1 with $n = 2$, the result is

$$\begin{aligned}
\int \cos^2(2x)\sin^2(2x)\,dx &= \frac{1}{2}\int \cos^2(u)\sin^2(u)\,du \\
&= \frac{1}{2}\left(\frac{\sin^3(u)\cos(u)}{4} + \frac{1}{4}\int \sin^2(u)\,du\right) \\
&= \frac{1}{2}\left(\frac{\sin^3(u)\cos(u)}{4} + \frac{1}{4}\left(\frac{u}{2} - \frac{\sin(u)\cos(u)}{2}\right)\right) \\
&= \frac{\sin^3(u)\cos(u)}{8} - \frac{\sin(u)\cos(u)}{16} + \frac{u}{16} + C \\
&= \frac{\sin^3(2x)\cos(2x)}{8} - \frac{\sin(2x)\cos(2x)}{16} + \frac{x}{8} + C.
\end{aligned}$$

Note: If you use SC3 rather than SC4, the answer will come out looking somewhat different but one form can be taken to the other via a judicious application of the trig identity $\sin^2(2x) + \cos^2(2x) = 1$. (C) For this one, use the trig substitution $x = 4\sin\theta$ so $dx = 4\cos\theta\,d\theta$. The integral becomes

$$\begin{aligned}
\int \frac{64\sin^2\theta\cos\theta\,d\theta}{4\cos\theta} &= 16\int \sin^2\theta\,d\theta \\
&= 16\cdot\left(\frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}\right) + C \quad \text{(SC1)} \\
&= 8\theta - 8\sin\theta\cos\theta + C \\
&= 8\sin^{-1}\left(\frac{x}{4}\right) - 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C \\
&= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C
\end{aligned}$$

(D) Rewrite as

$$\int \frac{x\ dx}{x^2+1} + 2\int \frac{dx}{x^2+1}$$

The first integral has the form $\int \frac{1}{2u}\ du$ for $u = x^2 + 1$. The second is the inverse tangent integral:

$$= \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x) + C.$$