MATH 136 – Calculus 2

Practice Day on Volumes of Solids of Revolution

March 11, 2020

Background

As we have seen in some examples, let the region R be defined by $0 \leq y \leq f(x)$ for all x in $[a, b]$. Then the volume of the solid of revolution obtained by rotating R about the x-axis is given by

$$\text{V (circle cross-sections)} = \int_a^b \pi (f(x))^2 \, dx \tag{1}$$

since the cross-section at x is a circle with radius $r = f(x)$. Note that this region "extends all the way to the x-axis" (the axis of rotation). If there is a "gap" between the region and the axis of rotation, then the cross-sections will be "washers" and we need to set up the volume integral differently:

$$\text{V (washer cross-sections)} = \int_a^b \pi \left((\text{outer radius})^2 - (\text{inner radius})^2 \right) \, dx \tag{2}$$

Today we want to practice on using (1) and (2) to set up and compute integrals for volumes of some regions of this type.

Questions

For each problem,

(i) Set up the volume integral for all of the following regions first.

(ii) The go back and evaluate each integral using the FTC part I. You may use a table of integrals anywhere here, as needed.

1. The solid generated by rotating the region between $y = x^4$ and the x-axis, $[a, b] = [0, 1]$.

   Solution: (Circle cross-sections) The volume is computed by

$$V = \int_0^1 \pi (x^4)^2 \, dx = \pi \frac{x^9}{9} \Big|_0^1 = \frac{\pi}{9}$$

2. The solid generated by rotating the region between $y = x^4$ and $y = x$, $[a, b] = [0, 1]$.

   Solution: (Washer cross-sections) Here the outer radius is x and the inner radius is x^4 since $x > x^4$ for all x with $0 < x < 1$. The volume is

$$V = \int_0^1 \pi(x^2 - x^8)\ dx = \pi \left(\frac{x^3}{3} - \frac{x^9}{9} \right) \Big|_0^1 = \frac{2\pi}{9}.$$

3. The solid generated by rotating the region between $y = \sec(x)$, and $y = \cos(x)$, $[a, b] = [0, \pi/4]$.

   Solution: (Washer cross-sections) The outer radius is $\sec(x)$ and the inner radius is $\cos(x)$. The volume is

$$\begin{aligned} V &= \int_0^{\pi/4} \pi(\sec^2(x) - \cos^2(x))\ dx \\ &= \pi \left(\tan(x) - \frac{x}{2} - \frac{\cos(x)\sin(x)}{2} \right) \Big|_0^{\pi/4} \\ &= \pi \left(1 - \frac{\pi}{8} - \frac{1}{4} \right) \\ &= \frac{6\pi - \pi^2}{8}. \end{aligned}$$

4. The solid generated by rotating the region between $y = (1 - x^2)^{1/4}$ and the x-axis, $[a, b] = [0, 1]$.

   Solution: (Circle cross-sections) The volume is

$$\int_0^1 \pi(1 - x^2)^{1/2}\ dx = \pi \left(\frac{x}{\sqrt{1 - x^2}} 2 + \frac{1}{2} \sin^{-1}(x) \right) \Big|_0^1 = \frac{\pi^2}{4}.$$

5. $f(x) = \sin(x)$, $[a, b] = [0, \pi]$.

   Solution: (Circle cross-sections) The volume is

$$V = \int_0^\pi \pi \sin^2(x)\ dx = \pi \left(\frac{x}{2} - \frac{\sin(x)\cos(x)}{2} \right) \Big|_0^\pi = \frac{\pi^2}{2}.$$