MATH 136 – Calculus 2
An Application of Integration By Parts:
Reduction Formulas
February 10, 2020

Background

There are a number of cases where repeated integration by parts can be used to completely evaluate integrals. Instead of working these out laboriously by hand every time, many mathematicians who need to use these will use a *reduction formula* that shows the general pattern of each application of the parts formula. These are usually obtained by consulting a compiled *table of integrals* rather than worked out by hand. Here is an example:

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

(This is derived by letting $u = x^n$ and $dv = e^{ax}\,dx$ as in an example from the video on "parts.") If we want to integrate $\int x^3 e^{4x}\,dx$, for instance, we apply this reduction formula three times with $n = 3$, then $n = 2$, then $n = 1$. The power of x is eventually reduced to a constant 1.

$$\begin{aligned}
\int x^3 e^{4x}\,dx &= \frac{x^3e^{4x}}{4} - \frac{3}{4}\int x^2 e^{4x}\,dx \\
&= \frac{x^3e^{4x}}{4} - \frac{3}{4}\left(\frac{x^2e^{4x}}{4} - \frac{2}{4}\int xe^{4x}\,dx\right) \\
&= \frac{x^3e^{4x}}{4} - \frac{3x^2e^{4x}}{16} + \frac{3}{8}\left(\frac{xe^{4x}}{4} - \frac{1}{4}\int e^{4x}\,dx\right) \\
&= \frac{x^3e^{4x}}{4} - \frac{3x^2e^{4x}}{16} + \frac{3xe^{4x}}{32} - \frac{3e^{4x}}{128} + C.
\end{aligned}$$

Questions

(A) Derive these reduction formulas for $n \geq 1$:

$$\int x^n\cos(ax)\,dx = \frac{x^n\sin(ax)}{a} - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx$$

and

$$\int x^n\sin(ax)\,dx = \frac{-x^n\cos(ax)}{a} + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx$$

by integrating by parts (once each).

(B) Using the two reduction formulas from part (A) in sequence, integrate:

$$\int x^2 \cos(3x)\ dx$$

(C) Derive the following reduction formula by applying parts with $u = \sin^{n-1}(x)$ and $dv = \sin(x)\ dx$:

$$\int \sin^n(x)\ dx = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n} \int \sin^{n-2}(x)\ dx.$$

(D) Use the formula from (C) to compute $\int \sin^5(x)\ dx$.

(E) Derive the following reduction formula for $n \geq 2$:

$$\int \tan^n(u)\ du = \frac{\tan^{n-1}(u)}{n-1} - \int \tan^{n-2}(u)\ du.$$

(Hint: This does *not* come from an integral by parts, but rather uses a "clever" application of the trig identity $\tan^2(x) = \sec^2(x) - 1$.) The case $n = 1$ is handled by a separate formula we have seen before (u-subsitution):

$$\int \tan(u)\ du = -\ln|\cos(u)| + C$$

(F) Use the formulas from (E) to integrate

$$\int \tan^5(x)\ dx$$