MATH 136 – Calculus 2
Practice Day on Properties of $\int_a^b f(x)\,dx$
January 27, 2020

Background

We have now introduced the definite integral $\int_a^b f(x)\,dx$ and discussed:

- the interpretation of of this number as a *signed area*,
- the *interval additivity property*: If f is integrable on $[a, b]$ and $a \le c \le b$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
- In one video for today, we also saw the *comparison property*: If f is integrable on $[a, b]$ and $m \le f(x) \le M$ for all x in $[a, b]$, then
$$m \cdot (b - a) \le \int_a^b f(x)\,dx \le M \cdot (b - a).$$

Questions Let $f(x)$ be the function on the interval $[0, 6]$ plotted on the back. Thinking of the definite integral as *signed area*, answer these questions.

(1) What are $\int_0^4 f(x)\,dx$, $\int_4^6 f(x)\,dx$, $\int_0^6 f(x)\,dx$? How can you get the last one by using the first two?

Answers: $\int_0^4 f(x)\,dx = 1 \cdot 2 + \frac{1}{2} \cdot (1 + 3) \cdot 2 = 6$. (I'm seeing this as a rectangle on $[0, 2]$ together with a trapezoid on $[2, 4]$, but there are other ways to do it as well.) $\int_4^6 f(x)\,dx = \frac{1}{2} \cdot \frac{3}{2} \cdot 3 - \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = 2$. (I'm seeing this as one triangle above the x-axis and a second triangle below the x-axis and doing the signed area. To get $\int_0^6 f(x)\,dx$, the best way is to use the interval additivity:
$$\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx = 6 + 2 = 8$$

(2) What is $\int_5^6 f(x)\,dx$? How can you tell?

Answer: It must be true that $\int_5^6 f(x)\, dx = 0$. The triangle above the x-axis for x in $[5, 11/2]$ is congruent to the triangle below the x-axis for x in $[11/2, 6]$, hence they have the same area. By the interval additivity property, the integral from 5 to 6 equals the integral from 5 to 11/2, plus the integral from 11/2 to 6. With the negative sign for the triangle below the x-axis, the sum is 0.

(3) Let m be the minimum value of $f(x)$ on $[3, 5]$ and let M be the maximum value of $f(x)$ on $[3, 5]$ (use the information from the graph). Find m, M and verify that

$$m \cdot (5 - 3) \leq \int_3^5 f(x)\, dx \leq M \cdot (5 - 3)$$

Answer: $m = 1$ and $M = 3$. We have $m \cdot (5-3) = 2$ and $M \cdot (5-3) = 6$. On the other hand

$$\int_3^5 f(x)\, dx = \int_3^4 f(x)\, dx + \int_4^5 f(x)\, dx = \frac{1}{2}(2+3) + \frac{1}{2}(3+1)$$

(two trapezoids). So this integral is $= \frac{9}{2}$. Note that $2 \leq \frac{9}{2} \leq 6$, so we have verified that the comparison property holds in this case.

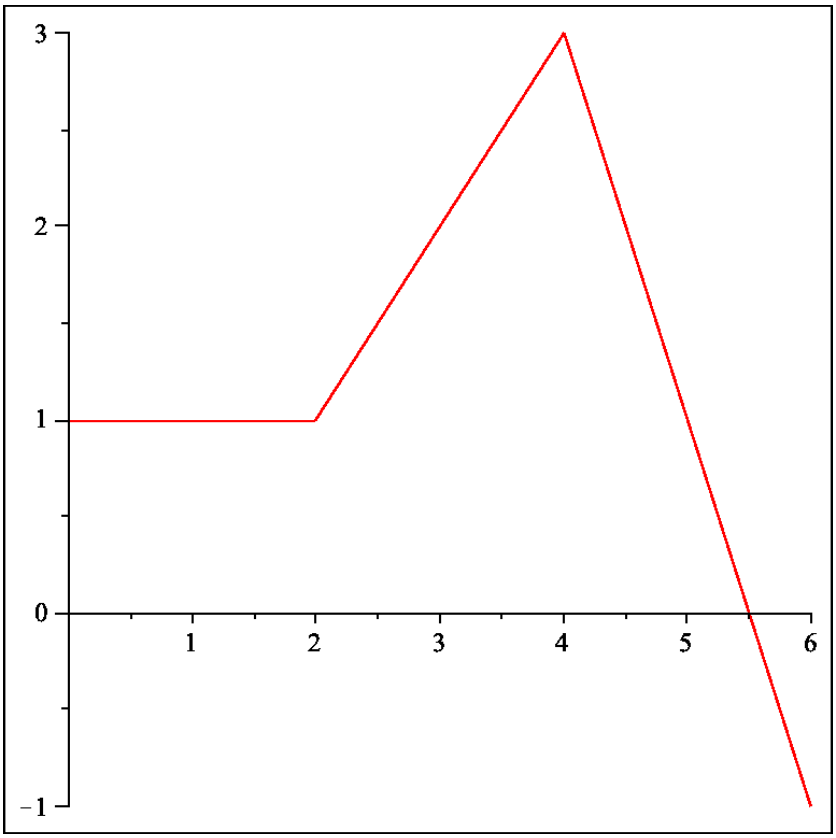

Figure 1: The region