MATH 136 – Calculus 2
Practice Day on the $L_N, R_N, M_N$ Riemann Sums
January 24, 2020

*Background*

We have now introduced the $L_N, R_N, M_N$ sum approximations to the area between the graph $y = f(x)$ and the $x$-axis, for $x$ in $[a, b]$. All three sums are based on a *partition*, or subdividision, of the interval $[a, b]$ into some given number $N$ of smaller intervals of width $\Delta x = \frac{b-a}{N}$. The endpoints of the smaller intervals are then given by

$$x_j = a + j\Delta x, \quad \text{for} \quad j = 0, 1, 2, \ldots, N.$$

The midpoint of the $j$th subinterval is

$$m_j = \frac{x_{j-1} + x_j}{2}.$$

Then we have

- using the left endpoints: $L_N = \sum_{j=1}^{N} f(x_{j-1})\Delta x$
- using the right endpoints: $R_N = \sum_{j=1}^{N} f(x_j)\Delta x$
- using the midpoints: $M_N = \sum_{j=1}^{N} f(m_j)\Delta x = \sum_{j=1}^{N} f\left(\frac{x_{j-1}+x_j}{2}\right)\Delta x.$

These are all special cases of the more general *Riemann sums*, named for the German Georg Friedrich Bernhard Riemann (1826 - 1866), who was one of the most celebrated and original mathematicians of the 19th century.

*Questions*

(A) Let $f(x) = x^2 + 3x + 1$ on the interval $[a, b] = [2, 4]$.

(1) Take $N = 3$ and compute the $L_3, R_3, M_3$ sums for this function.

*Answer:*

$$L_3 = \frac{886}{27} \doteq 32.81$$
$$R_3 = \frac{1210}{27} \doteq 44.81$$
$$M_3 = \frac{1042}{27} \doteq 38.59$$

(2) With the same function and the same interval, now take $N = 6$ and compute the $L_6, R_6, M_6$ sums for this function.

*Answer:*

$$L_6 = \frac{964}{27} \doteq 35.70$$
$$R_6 = \frac{1126}{27} \doteq 41.70$$
$$M_6 = \frac{2087}{54} \doteq 38.65$$

(3) Which is closer to the true area under the graph $y = x^2+3x+1$ on $[2, 4]$, $L_3$ or $L_6$? Explain by drawing the picture of the graph and the approximating rectangles.

*Answer:* The left-endpoint rectangles fit completely under the graph since $f(x) = x^2 + 3x + 1$ is increasing on the interval $[2, 4]$. This says that $L_3$ is strictly less than the actual area. However, as we go from $N = 3$ to $N = 6$, we pick up some of the missing area and $L_3 < L_6$. So $L_6$ is closer to the true area.

(4) Similarly, which is closer to the true area under the graph, $R_3$ or $R_6$?

*Answer:* The right-endpoint rectangles completely cover the area under the graph since $f(x) = x^2 + 3x + 1$ is increasing on the interval $[2, 4]$. This says that $L_3$ is strictly greater than the actual area. However, as we go from $N = 3$ to $N = 6$, we lose some of the extra area and $R_3 > R_6$. So $R_6$ is closer to the true area.

(5) Of the six numbers you computed in parts (1) and (2), which do you think is *the closest* to the true area under the graph? Explain.

*Answer:* It is probably the $M_6$ appoximation. Note that the midpoint rectangles will have area much closer to the area under the graph on that interval since the extra to the left of the midpoint is balanced by the missing area to the right of the midpoint.

(B) Repeat parts (1) and (2) of question A for $f(x) = \sin(x)$ on the interval $[a, b] = [0, \pi]$.

(1) Take $N = 3$ and compute the $L_3, R_3, M_3$ sums for this function.

*Answer:*

$$L_3 \doteq 1.9541$$
$$R_3 \doteq 1.9541$$
$$M_3 \doteq 2.0230$$

(2) With the same function and the same interval, now take $N = 6$ and compute the $L_6, R_6, M_6$ sums for this function.

*Answer:*

$$
\begin{aligned}
L_6 &\doteq 1.9451 \\
R_6 &\doteq 1.9451 \\
M_6 &\doteq 2.0230
\end{aligned}
$$

(These are exactly the same as for $N = 3$(!)) The answers to parts (3), (4), and (5) are not as easy to see in this case because the function $f(x) = \sin(x)$ is not *monotonic* on this interval–it is increasing on the first half, then decreasing on the second half. Note that we still expect the midpoint sums to be closer to the actual area, though.