MATH 136 – Calculus 2
Practice Days on Partial Fractions
February 17 and 18, 2020

Background

Every rational function (quotient of polynomials) can be written as *a polynomial plus a sum of one or more terms of the following forms*:

$$\frac{C}{(x+a)^k}, \qquad \frac{Ax+B}{(x^2+bx+c)^k},$$

where in the second case $x^2 + bx + c = 0$ has no real roots. A rational function expressed this way is said to be *decomposed into partial fractions*. The process of finding this decomposition is as follows: Given a rational function $\frac{f(x)}{g(x)}$,

1. First, if the degree of g is larger already, just proceed to step 2 with $\frac{f(x)}{g(x)}$. Otherwise, if the degree of $f(x)$ is greater than or equal to the degree of $g(x)$, *divide* $g(x)$ into $f(x)$ using polynomial long division and write $f(x) = q(x)g(x)+r(x)$ for some quotient $q(x)$ and remainder $r(x)$. Then
$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$
and the degree of $r(x)$ is less than the degree of $g(x)$. Continue to step 2 with $\frac{r(x)}{g(x)}$.

2. *Factor* $g(x)$ completely into a product of powers of linear polynomials and powers of quadratic polynomials with no real roots ("irreducible" quadratics). (The fact that this can always be done is one form of the so-called "Fundamental Theorem of Algebra". The famous mathematician and physicist Carl Friedrich Gauss gave the first complete proof of this result in 1799.)

3. Assemble the partial fractions: For each $(x + a)^m$ appearing in the factorization of $g(x)$, include a *group of terms*
$$\frac{C_1}{(x+a)} + \frac{C_2}{(x+a)^2} + \cdots + \frac{C_m}{(x+a)^m}$$

For each power $(x^2+bx+c)^n$ of an irreducible quadratic appearing in the factorization of $g(x)$, include a *group of terms*:

$$\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{(x^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(x^2+bx+c)^n}$$

4. Set the rational function from step 1 (either the original f/g or r/g as appropriate) equal to the sum of the partial fractions, clear denominators, and solve for the coefficients. This last step can be done either by substituting well-chosen x-values, or by equating coefficients of like powers of x on both sides and solving the resulting system of equations.

5. Integrate the polynomial quotient and the partial fractions. The integrals of the partial fractions of the form $\frac{C}{(x+a)^k}$ are easy u-substitution forms; the integrals of the partial fractions of the form $\frac{Ax+B}{(x^2+bx+c)^k}$ can be done with a combination of u- and trigonometric (tangent) substitutions. The most general form after integration contains a logarithm term plus an inverse tangent term.

Questions

1. $\displaystyle\int \frac{x^4+1}{x^3+25x}\,dx$

   Solution: The degree of the top is greater than the degree of the bottom, so we need to begin by dividing:

$$x^4+1=x(x^3+25x)+(-25x^2+1)$$

(that is, the quotient is $q(x)=x$ and the remainder is $r(x)=-25x^2+1$. Then, remembering that the eventual integral will include $\int x\,dx=\frac{x^2}{2}$, we proceed with the partial fractions on

$$\frac{-25x^2+1}{x^3+25x}=\frac{-25x^2+1}{x(x^2+25)}$$

By the general recipe above we want

$$\frac{-25x^2+1}{x(x^2+25)}=\frac{A}{x}+\frac{Bx+C}{x^2+25}$$

and this says

$$-25x^2 + 1 = A(x^2 + 25) + (Bx + C)x = (A + B)x^2 + Cx + 25A$$

after clearing denominators. Equating coefficients of powers of x, we have $25A = 1$, $C = 0$, $A + B = -25$, so $A = \frac{1}{25}$ and

$$B = -25 - \frac{1}{25} = \frac{-626}{25}.$$

Putting everything together and using a u-substitution on the final integral below, we have

$$\begin{aligned}\int \frac{x^4 + 1}{x^3 + 25x}\,dx &= \int x\,dx + \int \frac{1/25}{x}\,dx + \int \frac{(-626/25)x}{x^2 + 25}\,dx \\ &= \frac{x^2}{2} + \frac{1}{25}\ln|x| - \frac{313}{25}\ln|x^2 + 25| + C.\end{aligned}$$

(Note that if $C \neq 0$, in one of these integrals, the final answer would also contain an inverse tangent term.)

2. $\displaystyle\int \frac{x + 2}{x^2 + 4x + 3}\,dx$

Solution: We don't need to divide here. The bottom factors as $(x + 1)(x + 3)$, so the partial fractions have the form

$$\frac{x + 2}{x^2 + 4x + 3} = \frac{A}{x + 1} + \frac{B}{x + 3}$$

Clearing denominators we get

$$x + 2 = A(x + 3) + B(x + 1)$$

We could equate coefficients as in the first integral. But there is an easier way in cases like this where the bottom is a product of linear factors. Namely, this is supposed to be an equality of functions, valid for all real x. If we substitute $x = -1$ then we can isolate the term with A and solve for that coefficient: $(-1)+2 = A((-2)+3)$, so $A = \frac{1}{2}$. Similarly, with $x = -3$, we have $(-3) + 2 = B((-3) + 1)$, so $B = \frac{1}{2}$. Then

$$\begin{aligned}\int \frac{x + 2}{x^2 + 4x + 3}\,dx &= \int \frac{1/2}{x + 1}\,dx + \int \frac{1/2}{x + 3}\,dx \\ &= \frac{1}{2}\ln|x + 1| + \frac{1}{2}\ln|x + 3| + C.\end{aligned}$$

Using properties of logarithms, the integral can also be written as

$$\ln\left(\sqrt{|x^2+4x+3|}\right)+C,$$

do you see why?

3. $\displaystyle\int \frac{x^4+1}{x^4+3x^3+2x^2}\,dx$

Solution: This is another one where we need to divide first:

$$x^4+1=(1)(x^4+3x^3+2x^2)-3x^3-2x^2+1$$

so

$$\frac{x^4+1}{x^4+3x^3+2x^2}=1+\frac{-3x^3-2x^2+1}{x^4+3x^3+2x^2}.$$

We decompose the last term in partial fractions:

$$\frac{-3x^3-2x^2+1}{x^4+3x^3+2x^2}=\frac{-3x^3-2x^2+1}{x^2(x+1)(x+2)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+1}+\frac{D}{x+2}.$$

Clearing denominators:

$$-3x^3-2x^2+1=A(x+1)(x+2)+Bx(x+1)(x+2)+Cx^2(x+2)+Dx^2(x+1)$$

If we substitute $x=0$ we get $1=2A$, so $A=\frac{1}{2}$. Then with $x=-1$, we get $2=C$. With $x=-2$, we get $17=-4D$, so $D=\frac{-17}{4}$. Finally, from the coefficients of x^3 on both sides we get $-3=B+C+D$, so $-3=B+2-\frac{17}{4}$, so $B=\frac{-3}{4}$. This gives

$$\begin{aligned}\int \frac{x^4+1}{x^4+3x^3+2x^2}\,dx &= \int dx+\int\frac{1/2}{x^2}\,dx+\int\frac{-3/4}{x}\,dx\\ &\quad+\int\frac{2}{x+1}\,dx+\int\frac{-17/4}{x+2}\,dx\\ &= x-\frac{1}{2x}-\frac{3}{4}\ln|x|+2\ln|x+1|-\frac{17}{4}\ln|x+2|+C.\end{aligned}$$

4. $\displaystyle\int \frac{1}{x^4+5x^2+6}\,dx$ (Factor the bottom!)

Solution: The bottom factors as

$$(x^2+2)(x^2+3)$$

so the partial fractions look like

$$\frac{1}{x^4+5x^2+6} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+3}$$

Clearing denominators,

$$\begin{aligned} 1 &= (Ax+B)(x^2+3) + (Cx+D)(x^2+2) \\ &= (A+C)x^3 + (B+D)x^2 + (3A+2C)x + 3B + 2D \end{aligned}$$

Equating coefficients gives $A+C=0$ and $3A+2C=0$, so $A=C=0$ and $B+D=0, 3B+2D=1$, so $B=1$ and $D=-1$. The integral is

$$\int \frac{1}{x^2+2}\,dx - \int \frac{1}{x^2+3}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}(x/\sqrt{2}) - \frac{1}{\sqrt{3}}\tan^{-1}(x/\sqrt{3}) + C$$