MATH 136 – Calculus 2
Practice Days on Partial Fractions
February 17 and 18, 2020

Background

Every rational function (quotient of polynomials) can be written as *a polynomial plus a sum of one or more terms of the following forms*:

$$\frac{C}{(x+a)^k}, \qquad \frac{Ax+B}{(x^2+bx+c)^k},$$

where in the second case $x^2 + bx + c = 0$ has no real roots. A rational function expressed this way is said to be *decomposed into partial fractions*. The process of finding this decomposition is as follows: Given a rational function $\frac{f(x)}{g(x)}$,

1. First, if the degree of g is larger already, just proceed to step 2 with $\frac{f(x)}{g(x)}$. Otherwise, if the degree of $f(x)$ is greater than or equal to the degree of $g(x)$, *divide* $g(x)$ into $f(x)$ using polynomial long division and write $f(x) = q(x)g(x)+r(x)$ for some quotient $q(x)$ and remainder $r(x)$. Then
$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$
and the degree of $r(x)$ is less than the degree of $g(x)$. Continue to step 2 with $\frac{r(x)}{g(x)}$.

2. *Factor* $g(x)$ completely into a product of powers of linear polynomials and powers of quadratic polynomials with no real roots ("irreducible" quadratics). (The fact that this can always be done is one form of the so-called "Fundamental Theorem of Algebra". The famous mathematician and physicist Carl Friedrich Gauss gave the first complete proof of this result in 1799.)

3. Assemble the partial fractions: For each $(x + a)^m$ appearing in the factorization of $g(x)$, include a *group of terms*
$$\frac{C_1}{(x+a)} + \frac{C_2}{(x+a)^2} + \cdots + \frac{C_m}{(x+a)^m}$$

For each power $(x^2+bx+c)^n$ of an irreducible quadratic appearing in the factorization of $g(x)$, include a *group of terms*:

$$\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{(x^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(x^2+bx+c)^n}$$

4. Set the rational function from step 1 (either the original f/g or r/g as appropriate) equal to the sum of the partial fractions, clear denominators, and solve for the coefficients. This last step can be done either by substituting well-chosen x-values, or by equating coefficients of like powers of x on both sides and solving the resulting system of equations.

5. Integrate the polynomial quotient and the partial fractions. The integrals of the partial fractions of the form $\frac{C}{(x+a)^k}$ are easy u-substitution forms; the integrals of the partial fractions of the form $\frac{Ax+B}{(x^2+bx+c)^k}$ can be done with a combination of u- and trigonometric (tangent) substitutions. The most general form after integration contains a logarithm term plus an inverse tangent term.

Questions

1. We'll work out $\displaystyle\int \frac{x^4+1}{x^3+25x}\,dx$ together

2. $\displaystyle\int \frac{x+2}{x^2+4x+3}\,dx$

3. $\displaystyle\int \frac{x^4+1}{x^4+3x^3+2x^2}\,dx$

4. $\displaystyle\int \frac{1}{x^4+5x^2+6}\,dx$ (Factor the bottom!)