MATH 136 – Calculus 2
Normal Probabilities and Tabulated Values
March 18, 2020

Background

The standard normal pdf is the function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The factor $1/\sqrt{2\pi}$ makes the total area equal to 1 as we require for a pdf, but this is not easy to show given what we have learned to this point. Probabilities for a standard normal random variable (i.e. normal distribution with $\mu = 0$, $\sigma = 1$) are given in the table on the accompanying sheet.

In today's problems, you will learn more about normal pdfs and practice using the table to answer questions about normally distributed quantities.

Questions

A) The standard normal pdf is a function that has *no elementary antiderivative.* This means that the entries in the table on the back were computed by approximate numerical integration techniques. Approximate the area given by

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

using a midpoint Riemann sum with $N = 5$. How close is your value to the table value for $z = 1.0$? What would you need to do to get a more accurate value?

Solution: The value of the Riemann sum is $\doteq .3417$. The table gives .3413. This is very close (error is .0004). To get a more accurate value, you would need to use a bigger N, or use a more accurate method like Simpson's Rule (see lab on numerical integration).

B) The standard normal pdf above is just one of a 2-parameter family of normal pdfs give by the formula

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

The parameter μ shifts the center of the distribution, and the σ controls the width of the peak. Show using the subsitution $u = \frac{x-\mu}{\sigma}$ that

$$\int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}\, dx = \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

In other words, areas under *any normal pdf* graph can be computed from the table for the standard normal by taking the original limits of integration a, b and "standardizing" to the new limits

$$\frac{a-\mu}{\sigma}, \quad \frac{b-\mu}{\sigma}$$

for the standard normal.

Solution: For $u = \frac{x-\mu}{\sigma}$, we have $du = \frac{1}{\sigma} dx$, so $dx = \sigma\, du$. Converting the limits of integration as well, if $x = a$, then $u = (a-\mu)/\sigma$ and if $x = b$, then $u = (b-\mu)/\sigma$. Therefore

$$\int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}\, dx = \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

as claimed. (Note that the σ from the dx cancels the $1/\sigma$ from the constant multiplying the exponential in the first integral.)

C) Let Z be a standard normal.

1) Find $P(-2.13 < Z < -0.56)$

Solution: By symmetry, this is the same as

$$\begin{aligned} P(0.56 < X < 2.13) &= P(0 < X < 2.13) - P(0 < X < 0.56) \\ &= .4834 - .2123 \\ &= .3711 \end{aligned}$$

2) Find c such that $P(Z > c) = .05$

Solution: This is approximately $c = 1.645$ (The closest entries from the table are $z = 1.64$ giving area .4495 and $z = 1.65$ giving area .4505, so c is roughly halfway between 1.64 and 1.65.)

D) Let Y be normally distributed with mean $\mu 6$ and $\sigma = 2$. Find

1) $P(6 < Y < 7)$

Solution: Using the idea from B above, this is the same as

$$P\left(\frac{6-6}{2} < Z < \frac{7-6}{2}\right) = P(0 < Z < .5) = .1915$$

from the standard normal table.

2) $P(7 < Y < 8)$

Solution: Standardizing, $\frac{7-6}{2} = .5$ and $\frac{8-6}{2} = 1$. Then

$$\begin{aligned} P(0.5 < Z < 1) &= P(0 < Z < 1) - P(0 < Z < 0.5) \\ &= .3413 - .1915 \\ &= .1498. \end{aligned}$$

E) Small packages of Oyster brand chowder crackers have a stated label weight of 45 grams. The actual weights of the packages are normally distributed with mean $\mu = 46.3$ grams and $\sigma = .7$ grams. Let Y be the weight of a single package selected at random from the production line. What is the probability $P(Y > 46)$?

Solution:

$$P(Y > 46) = P\left(Z > \frac{46 - 46.3}{.7}\right) = P(Z > -.43)$$

To find this area from the table, note by symmetry that this is the same as

$$\begin{aligned} P(Z > 0) + P(-.43 < Z < 0) &= P(Z > 0) + P(0 < Z < .43) \\ &= .5 + .1664 \\ &= .6664 \end{aligned}$$

(The company could probably not fill the packages this full and still meet the stated label weight!)