MATH 136 – Calculus 2
Practice on Separable Differential Equations and Applications
March 20, 2018

Background

A differential equation is a relation between derivative(s) of an unknown function and other known functions. A differential equation often serves as a *mathematical model* for how some quantity is evolving with respect to time. For example, Newton's Law of Cooling (and Heating) says: *if an object is placed into a surrounding medium held at constant temperature A, then the object's temperature changes at a rate proportional to the difference between its temperature and the ambient temperature.* If T is the object's temperature, this statement is equivalent to the differential equation

$$\frac{dT}{dt} = k(T - A)$$

for some constant k.

Any differential equation of the form

$$\frac{dy}{dx} = g(y) \cdot h(x)$$

is called a "separable" equation and can be solved by

- *separating the variables* to the form
$$\frac{dy}{g(y)} = h(x)\ dx$$
- *integrating on both sides* (on the left, treat the variable of integration as y, not x; this can be justified by the substitution method for integration):
$$\int \frac{dy}{g(y)} = \int h(x)\ dx$$
- *solving the resulting equation for y*

(A) To practice, solve the following separable equations:

(1)

$$\frac{dy}{dx} = \frac{y}{x^2+1}$$

Solution: Separated:

$$\frac{dy}{y} = \frac{1}{x^2+1}\ dx$$

We integrate both sides:

$$\int \frac{dy}{y} = \int \frac{1}{x^2+1}\ dx \Rightarrow \ln|y| = \tan^{-1}(x) + C$$

To solve for y, we exponentiate, then take both choices of sign:

$$y = \pm e^C e^{\tan^{-1}(x)} = ke^{\tan^{-1}(x)}$$

where k is a positive or negative constant: $k = \pm e^C$.

(2)

$$\frac{dy}{dx} = x^3y^2 + y^2$$

(Hint: factor on the right, then you can separate variables)

Solution: Factored, $\frac{dy}{dx} = y^2(x^3+1)$, so

$$\int \frac{dy}{y^2} = \int x^3 + 1\ dx$$

and hence

$$\frac{-1}{y} = \frac{x^4}{4} + x + C$$

Then solving for y,

$$y = \frac{-1}{x^4/4 + x + C}$$

(B) Solve the following Newton's Law of Cooling problem: A hot cup of coffee is poured at time $t = 0$ with the temperature being 80° C. The cup is placed on a desk in a room with temperature maintained at 23° C. Five minutes later, the coffee has cooled to 70° C. At what time will the coffee have cooled down to 40° F?

Solution: Here the ambient temperature is $A = 23$ degrees C. From the explanation above, we know the temperature T of the cup of coffee satisfies

$$\frac{dT}{dt} = k(T - 23)$$

This separable:

$$\int \frac{dT}{T - 23} = \int k dt$$

so

$$\ln |T - 23| = kt + C$$

and

$$T = 23 + De^{kt}$$

for some positive or negative constant $D = \pm e^C$. When $t = 0$ we know $T = 80$, so $80 = 23 + De^{k \cdot 0} = 23 + D$. Hence $D = 57$. Then when $t = 5$, we have $70 = 23 + 57e^{5k}$, so $e^{5k} = \frac{47}{57}$ and $k = \frac{\ln(47/57)}{5} \doteq -.0386$. Finally we need to solve the equation

$$40 = 23 + 57e^{(-.0386)t}$$

for t. The solution is

$$t = \frac{\ln(17/57)}{-.0386} \doteq 31.4 \text{ minutes}$$