MATH 136 – Calculus 2
Arc Length
March 13, 2020

Background We have now seen why the arc length of a graph $y = f(x)$ for $x = a$ to $x = b$ can be computed by the integral

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

It is actually relatively rare that these integrals can be computed exactly using the FTC. Many of them require numerical approximations because the complicated integrand tends to have no elementary antiderivative. The following cases are a few of the ones that *can be* evaluated by the FTC, either because the $\sqrt{1 + (f'(x))^2}$ is rather simple, or because the $1 + (f'(x))^2$ is set up to be a perfect square(!)

Questions

1. Find the arc length of the curve $y = 4 + x^{3/2}$ for $x = 0$ to $x = 4$.

   Solution: With $f(x) = 4 + x^{3/2}$, $f'(x) = \frac{3x^{1/2}}{2}$ and the arc length integral is
   $$L = \int_0^4 \sqrt{1 + \left(\frac{3x^{1/2}}{2}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9x}{4}}\, dx$$
   We can evaluate this by letting $u = 1 + \frac{9x}{4}$ and $dx = \frac{4}{9}\, du$.
   $$L = \int_{u=1}^{u=10} \frac{4}{9} u^{1/2}\, du = \frac{8}{27} u^{3/2}\Big|_1^{10} = \frac{8}{27}(10\sqrt{10} - 1).$$

2. Find the arc length of the curve $y = \frac{x^{3/2}}{3} - x^{1/2}$ for $x = 2$ to $x = 8$.

   Solution: Here for $f(x) = \frac{x^{3/2}}{3} - x^{1/2}$, we have $f'(x) = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$. When we compute the quantity inside the square root in the arc length integrand a sort of algebraic miracle happens:
   $$1 + (\frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2})^2 = 1 + \frac{x}{4} - \frac{1}{2} + \frac{1}{4x} = \frac{x}{4} + \frac{1}{2} + \frac{1}{4x}$$

is also the perfect square $\left(\frac{x^{1/2}}{2} + \frac{x^{-1/2}}{2}\right)^2$. So we must compute

$$\begin{aligned}\int_2^8 \frac{x^{1/2}}{2} + \frac{x^{-1/2}}{2}\,dx &= \frac{x^{3/2}}{3} + x^{1/2}\Big|_2^8 \\ &= \frac{11\sqrt{8}}{3} - \frac{5\sqrt{2}}{3} \\ &= \frac{17\sqrt{2}}{3}\end{aligned}$$

(using $\sqrt{8} = 2\sqrt{2}$ to simplify).

3. The most important example of this type from a practical point of view is probably the following. Chains or wires hanging under their own weight suspended from their endpoints have the shape of a curve called the *catenary*. Up to scaling, a catenary can always be given by an function of the form

$$f(x) = \frac{e^x + e^{-x}}{2}.$$

This function is also called the *hyperbolic cosine* and there are several sections in our textbook dealing with these and related functions. Find the arc length of the catenary curve above for $x = -1$ to $x = 1$.

Solution: This one is similar to the previous one in that when we compute $1 + (f'(x))^2$ for $f(x) = \frac{e^x+e^{-x}}{2}$, the result is also a perfect square:

$$1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2.$$

Hence the arc length integral is

$$\begin{aligned}\int_{-1}^1 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\,dx &= \int_{-1}^1 \frac{e^x + e^{-x}}{2}\,dx \\ &= \frac{e^x - e^{-x}}{2}\Big|_{-1}^1 \\ &= \frac{e - e^{-1}}{2} - \frac{e^{-1} - e}{2} \\ &= e - e^{-1}.\end{aligned}$$