College of the Holy Cross
Math 135, Section 1 – Solutions for Midterm Exam 2
Friday, November 1

I. Compute the indicated limits. You must show all necessary work to justify your answer to receive full credit.

(a) (5) $\displaystyle\lim_{x\to 1}\frac{x^2-5x+2}{2x^2-2x+3}$

Solution: The top is going to -2 and the bottom is going to 3 as $x\to 1$. By the Limit Quotient Rule, the limit is $=-2/3$.

(b) (5) $\displaystyle\lim_{x\to 1}\frac{x^2-7x+6}{2x^2-2x}$

Solution: This is a $0/0$ indeterminate limit so we should factor the top and bottom and try to cancel:

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-7x+6}{2x^2-2x} &= \lim_{x\to 1}\frac{(x-6)(x-1)}{2x(x-1)}\\ &= \lim_{x\to 1}\frac{x-6}{2x}\\ &= -5/2.\end{aligned}$$

(c) (5) $\displaystyle\lim_{x\to\infty}\frac{x^2-5x+2}{2x^2-2x+3}$

Solution: Divide the top and bottom by x^2 then take the limit as $x\to\infty$:

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2-5x+2}{2x^2-2x+3} &= \lim_{x\to\infty}\frac{1-5/x+2/x^2}{2-2/x+3/x^2}\\ &= 1/2.\end{aligned}$$

(d) (5) $\displaystyle\lim_{\theta\to 0}\frac{\sin(4\theta)}{\sin(5\theta)}$

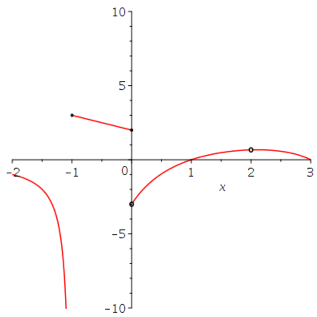
Solution: Since $\lim_{x\to 0}\frac{\sin(x)}{x}=1$, this limit equals

$$\lim_{\theta\to 0}\frac{4}{5}\,\frac{\frac{\sin(4\theta)}{4\theta}}{\frac{\sin(5\theta)}{5\theta}}=\frac{4}{5}.$$

1. The graph of the function

$$f(x)=\begin{cases}\frac{1}{x+1} & x<-1\\ -x+2 & -1\le x\le 0\\ \frac{4x^3-24x^2+44x-24}{x^3-5x^2+2x+8} & 0<x<3\end{cases}$$

is shown in Figure 1.



(a) (10) What are $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$? (In your answer say clearly which is which.)

Solution: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x + 2 = 2$ and

$$\lim_{x \to 0^+} = \lim_{x \to 0^+} \frac{4x^3 - 24x^2 + 44x - 24}{x^3 - 5x^2 + 2x + 8} = -3$$

(These can also be estimated from the graph.

(b) (15) Find all x in $(-2, 3)$ where f is discontinuous. Give the types of each of the discontinuities.

Solution: There are *three*: $x = -1$ is an *infinite discontinuity* since $\lim_{x \to -1^-} \frac{1}{x+1} = -\infty$. $x = 0$ is a *jump discontinuity* since the one-sided limits exist but are unequal. Finally, $x = 2$ is a *removable discontinuity.* Note the open circle at $x = 2$ in the graph. This is a reflection of the fact that the function

$$\frac{4x^3 - 24x^2 + 44x - 24}{x^3 - 5x^2 + 2x + 8}$$

gives $0/0$ at $x = 2$, but the limit as $x \to 2$ does exist and equals $2/3$.

2. Do not use the short-cut differentiation rules from Chapter 3 in this question.

(a) (5) State the limit definition of the derivative $f'(x)$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(provided that the limit exists).

(b) (10) Estimate the derivative of $f(x) = \sqrt{x+2}$ at $a = 7$ *numerically* by computing difference quotients of f with $h = \pm.1$, then $h = \pm.01$. Enter your values in the table below, and then state what your estimate of $f'(7)$ is.

Solution:

h	$-.1$	$-.01$	$.01$	$.1$
difference quotient value	.1671	.1667	.1666	.1662

$f'(6) \doteq \underline{.1666}$

(c) (10) Use the definition to compute the derivative function of $f(x) = \sqrt{x+2}$.

Solution: Multiply by the conjugate radical, simplify, and take the limit:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \\
&= \lim_{hto0} \left(\frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}\right) \cdot \left(\frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}\right) \\
&= \lim_{h\to 0} \frac{(x+h+2)-(x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2} + \sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

(d) (5) Find the equation of the line tangent to the graph $y = \sqrt{x+2}$ at $a = 7$.

Solution: The point on the graph is $(7, 3)$. The slope of the tangent line is $f'(7) = \frac{1}{6}$ (You could also use the estimated value $f'(x) \doteq .1666$ from part b.) The equation of the tangent line is

$$y - 3 = \frac{1}{6}(x - 7),$$

or

$$y = \frac{1}{6}x + \frac{11}{6}$$

in slope-intercept form.

3. Use the short-cut rules to compute the following derivatives. You may use any correct method, but you must show work for full credit.

   (a) (5) $\dfrac{d}{dx}\left(\dfrac{3}{\sqrt{x}} - e^x + 3x\right)$

   (b) *Solution:* First rewrite the function using rules for exponents as

   $$3x^{-1/2} - e^x + 3x$$

   Then the derivative is

   $$\frac{-3}{2}x^{-3/2} - e^x + 3$$

   (c) (10) $\dfrac{d}{dv}\left((v^2 - 2v)(v^3 + 1)\right)$

   (d) *Solution:* By the product rule, the derivative is

   $$(v^2 - 2v)(3v^2) + (v^3 + 1)(2v - 2).$$

   This form is OK by the directions. The simplified form is

   $$5v^4 - 8v^3 + 2v - 2.$$

   (Note: This could also be done by multiplying out the product, then differentiating term by term using the power rule.)

   (e) (10) $\dfrac{d}{dx}\left(\dfrac{x^2 - 4x + 3}{x^4 - x}\right)$

   (f) *Solution:* By the quotient rule, the derivative is

   $$\frac{(x^4 - x)(2x - 4) - (x^2 - 4x + 3)(4x^3 - 1)}{(x^4 - x)^2}.$$

   Again, by the directions, this form is OK; the simplified form is

   $$\frac{-2x^5 + 12x^4 - 12x^3 - x^2 + 3}{(x^4 - x)^2}.$$