**Holy Cross College, Fall Semester, 2019**
**MATH 135, Section 01, Final Exam B Solutions**
**Saturday, December 21, 8:00 AM**

1. [20 points] One of the functions given in the following table is linear and the other is exponential. Find a formula for the *exponential* one and place it in the appropriate box. In the box for the other one, write “Linear.”

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x)$ | 1.2 | 2.4 | 4.8 | 9.6 | 19.2 |
| $g(x)$ | $-3.3$ | $-4.4$ | $-5.5$ | $-6.6$ | $-7.7$ |

*Answer:*

$$\begin{aligned} f(x) &= 1.2 \cdot 2^{x-1} = 0.6 \cdot 2^x \\ g(x) &= \text{Linear} \end{aligned}$$

2. Let $f(x) = x^3 + 3x + 1$ and $g(x) = \sqrt{x^2+3}$.

(a) [10 points] Find the formula for the composition $g(f(x))$

*Answer:*

$$g(f(x)) = \sqrt{(x^3+3x+1)^2+3}$$

(b) [10 points] Write $g(x) = h(k(x))$ for two other functions $h$ and $k$, where

*Answer:* One possibility is $h(x) = \sqrt{x}$ and $k(x) = x^2 + 3$.

(c) [10 points] Find the average rate of change of $g(x)$ per unit change in $x$ on the interval $[0, 2]$.

*Answer:* The average rate of change is

$$\frac{g(2) - g(0)}{2-0} = \frac{\sqrt{7} - \sqrt{3}}{2} \doteq .457.$$

3. Compute the following limits [10 points each]. *Any legal method is OK.*

(a) (*) $\lim_{x \to 0} (1+5x)^{1/x}$

*Answer:* This is a $1^\infty$ indeterminate form. Take logarithms, rearrange to 0/0 form, then apply L’Hopital’s Rule. The answer is $e^5$.

(b) $\lim_{x\to 3} \dfrac{x^2-6x+9}{x^2-5x+6}$

This is a 0/0 indeterminate form. It can be evaluated by factoring the top and the bottom and cancelling a factor of $x-3$:

$$\lim_{x\to 3} \frac{(x-3)^2}{(x-3)(x-2)} = \lim_{x\to 3} \frac{x-3}{x-2} = 0.$$

Alternatively, one can also use L'Hopital's Rule.

(c) $\lim_{x\to 0} \dfrac{\sin(3x)}{x}$

*Answer:*

$$= \lim_{x\to 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3.$$

4.

(a) [10 points] State the limit definition of the derivative:

*Answer:*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided that the limit exists.

(b) [10 points] Use the definition to compute $f'(x)$ for $f(x) = \dfrac{1}{x^2}$.

*Answer:* We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\ &= \lim_{h\to 0} \frac{(x^2-(x+h)^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0} \frac{-2xh-h^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0} \frac{-2x-h^2}{(x+h)^2x^2} \\ &= \frac{-2x}{x^4} = \frac{-2}{x^3} \end{aligned}$$

(Note that this agrees with the shortcut rule for differentiating $f(x) = x^{-2}$.

(c) [10 points] Find the equation of the tangent line to the graph $y = \dfrac{1}{x^2}$ at the point $(2, 1/4)$. Note: you can do this one even if you were not able to complete part b above.

*Answer:* The slope is $\frac{-2}{2^3} = \frac{-1}{4}$. So by the point-slope form, the equation of the tangent line is

$$y - 1/4 = \frac{-1}{4}(x-2)$$

5. Compute the following derivatives using the derivative rules. You need not simplify.

(a) [10 points] $f(z) = z^5 - \frac{1}{\sqrt[6]{z}} + e^z$.

*Answer*

$$f'(z) = 5z^4 + \frac{1}{6}x^{-7/6} + e^z.$$

(b) [10 points] $g(x) = (x^3+1)(x^2-1)$

*Answer:* By the product rule,

$$g'(x) = (x^3+1)(2x) + (x^2-1)(3x^2).$$

This can also be done by multiplying out the product, then differentiating.

(c) (*) [10 points] $h(t) = \frac{\ln(t^2+3)}{\sin^{-1}(t)}$

*Answer:* By the quotient rule and the rules for the natural Log and inverse sine,

$$h'(t) = \frac{\sin^{-1}(t) \cdot \frac{2t}{t^2+3} - \ln(t^2+3) \cdot \frac{1}{\sqrt{1-t^2}}}{(\sin^{-1}(t))^2}.$$

(d) (*) [10 points] Find $y' = \frac{dy}{dx}$ if $3x^3y^2 - 7y^3 + \tan(x) = y$.

*Answer:* Differentiating implicitly,

$$\frac{dy}{dx} = \frac{-\sec^2(x) - 9x^2y^2}{6x^3y - 21y^2 - 1}$$

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6. (*) [5 points each] All parts of this question refer to the function $f(x)$ whose *derivative* $f'(x)$ is plotted in Figure 1 above. Assume the domain of $f$ contains only the interval shown in the plot.

(a) On which interval(s) is $f(x)$ increasing?

*Answer:* Increasing on $(1, 4)$.

(b) On which interval(s) is $y = f(x)$ concave down?

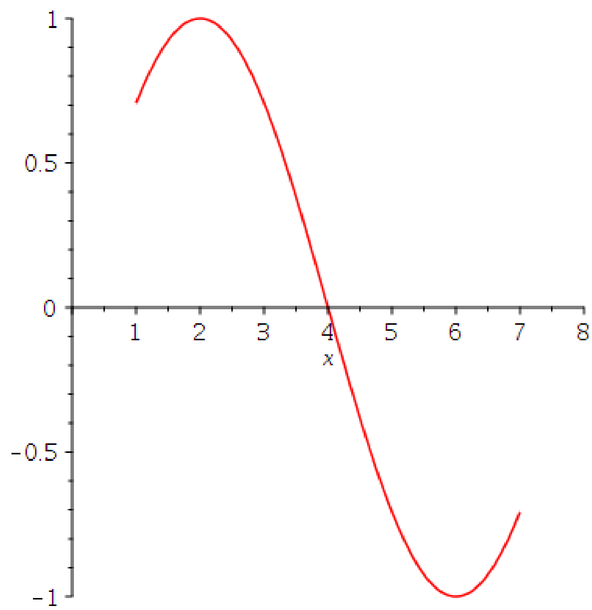



Figure 1: Plot for question 6.

*Answer:* On $(2, 6)$.

(c) How many critical points does $f$ have? Classify them as local maxima, local minima or neither.

*Answer:* One critical point at $x = 4$. This is a local maximum for $f(x)$ by the first derivative test.

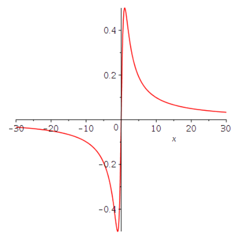
7. (*) [20 points] If the ticket price for the upcoming "Star Wars" movie is set at \$10, then 1000 tickets for the first showing will be sold. However, for each \$0.25 increase in the ticket price, the number of tickets sold will go down by 10 tickets. Each person who buys a ticket will also purchase \$12 of candy, popcorn and drinks at the concession stand. What is the maximum revenue that can be earned earned by the theater from the ticket and concession sales?

*Answer:* Let $x$ be the number of \$0.25 increases over the \$10 level. Then with the ticket price at $10 + 0.25x$, $1000 - 10x$ tickets will be sold, yielding revenue of $(1000 - 10x)(10 + 0.25x)$. In addition, $12(1000 - x)$ dollars in candy, popcorn, and drinks will be sold, yielding total revenue of

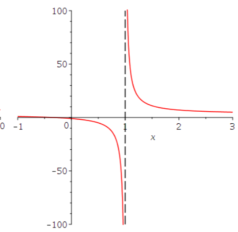
$$R(x) = (1000 - 10x)(10 + 0.25x) + 12(1000 - 10x) = 22000 + 30x - 2.5x^2$$

We differentiate and set equal to zero to find critical points:

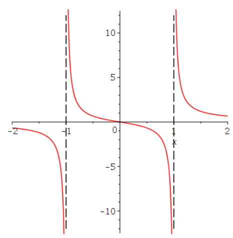
$$R'(x) = 30 - 5x = 0$$



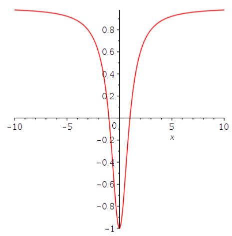
(a) Plot I



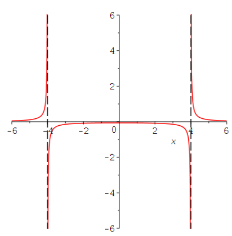
(b) Plot II



(c) Plot III



(d) Plot IV



(e) Plot V

When $x = 6$, and $R''(x) = -5 < 0$, so this is a local and global maximum for the revenue. The ticket price should be $\$(10 + 0.25 \cdot 6) = \$11.50$ and the maximum revenue is $R(6) = \$22090$.

8. (*) [3 points each] By considering the location of vertical and horizontal asymptotes, $x$- and $y$-axis intercepts, etc. determine which of the following functions matches each graph. Circle the number of the graph showing each of the following functions.

(a) $f(x) = \dfrac{1}{x^2 - 16}$

*Answer:* V (vertical asymptotes at $x = \pm 4$, horizontal asymptote at $y = 0$)

(b) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

*Answer:* IV (no vertical asymptote and horizontal asymptote at $y = 1$)

(c) $f(x) = \dfrac{x}{x^2 + 1}$

*Answer:* I (no vertical asymptotes, $x$-axis intercept at $x = 0$, horizontal asymptote at $y = 0$.)

(d) $f(x) = \dfrac{x}{x^2 - 1}$

*Answer:* III (vertical asymptotes at $x = \pm 1$, $x$-axis intercept at $x = 0$, horizontal asymptote at $y = 0$.)

(e) $f(x) = \dfrac{3x + 1}{x - 1}$

*Answer:* II (only one vertical asymptote at $x = 1$)