Holy Cross College, Fall Semester, 2019
MATH 135, Section 01, Final Exam A Solutions
Saturday, December 21, 8:00 AM

1. [20 points] One of the functions given in the following table is linear and the other is exponential. Find a formula for the *linear* one and place it in the appropriate box. In the box for the other one, write "Exponential."

x	1	2	3	4	5
$f(x)$	1.2	2.4	4.8	9.6	19.2
$g(x)$	-3.3	-4.4	-5.5	-6.6	-7.7

Answer:

$$\begin{aligned} f(x) &= \text{Exponential} \\ g(x) &= -1.1(x-1) - 3.3 = -1.1x - 2.2 \end{aligned}$$

2. Let $f(x) = x^3 + 3x + 1$ and $g(x) = \sqrt{x^2+3}$.

(a) [10 points] Find a formula for the composition $f(g(x))$.

Answer: $f(g(x)) = (x^2+3)^{3/2} + 3(x^2+3)^{1/2} + 3$.

(b) [10 points] Write $g(x) = h(k(x))$ for two other functions h and k, where $h(x) \neq x$ and $k(x) \neq x$.

Answer: $h(x) = \sqrt{x}$ and $k(x) = x^2 + 3$ is one possibility.

(c) [10 points] Find the average rate of change of $f(x)$ per unit change in x on the interval $[1, 3]$.

Answer: The average rate of change is

$$\frac{f(3) - f(1)}{3-1} = \frac{37-5}{2} = 16.$$

3. Compute the following limits [10 points each]. *Any legal method is OK.*

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{x}$

Answer: This limit is

$$\lim_{x\to 0} 4 \cdot \frac{\sin(4x)}{4x} = 4 \cdot 1 = 4.$$

(b) $\lim_{x\to 2} \frac{x^2-8x+12}{x^2-2x}$

Answer: This is a 0/0 indeterminate form. It can be done either by factoring the top and the bottom and canceling a factor of $x-2$ get

$$\lim_{x\to 2} \frac{(x-2)(x-6)}{x(x-2)} = \lim_{x\to 2} \frac{x-6}{x} = -2$$

or by using L'Hopital's Rule.

(c) (*) $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$

Answer: This is a 1^∞ indeterminate form. Take logarithms, rearrange to 0/0 form, then apply L'Hopital's Rule. The answer is e^3.

4.

(a) [10 points] State the limit definition of the derivative:

Answer:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided that the limit exists.

(b) [10 points] *Use the definition* to compute $f'(x)$ for $f(x) = \frac{1}{x+1}$.

Answer: We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\ &= \lim_{h\to 0} \frac{(x+1)-(x+h+1)}{h(x+h+1)(x+1)} \\ &= \lim_{h\to 0} \frac{-h}{h(x+h+1)(x+1)} \\ &= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} \\ &= \frac{-1}{(x+1)^2} \end{aligned}$$

(Note that this agrees with the result obtained from the chain rule for the derivative of $f(x) = (x+1)^{-1}$.)

(c) [10 points] Find the equation of the tangent line to the graph $y = frac1x + 1$ at the point $(2, 1/3)$. Note: you can do this one even if you were not able to complete part b above.

Answer: The slope is $f'(2) = \frac{-1}{9}$ so the equation is

$$y - \frac{1}{3} = \frac{-1}{9}(x-2)$$

by the point-slope form.

5. Compute the following derivatives using the derivative rules. You need not simplify.

(a) [10 points] $f(x) = x^{4/5} - \frac{1}{\sqrt[3]{x}} + e^x$.

Answer:

$$f'(x) = \frac{4}{5}x^{-1/5} + \frac{1}{3}x^{-4/3} + e^x$$

(b) [10 points] $g(x) = \frac{x^2+1}{x^4+1}$

Answer: By the quotient rule,

$$g'(x) = \frac{(x^4+1)(2x) - (x^2+1)(4x^3)}{(x^4+1)^2}$$

(c) (*) [10 points] $h(s) = \ln(4s^2+2)\tan^{-1}(s)$

Answer: By the product and chain rules,

$$h'(s) = \ln(4s^2+2) \cdot \frac{1}{s^2+1} + \tan^{-1}(s) \cdot \frac{8s}{4s^2+2}.$$

(d) (*) [10 points] Find $y' = \frac{dy}{dx}$ if $4x^2y^3 - 2y^2 + \sin(x) = x^4$.

Answer: Differentiating implicitly,

$$\frac{dy}{dx} = \frac{-\cos(x) + 4x^3 - 8xy^3}{12x^2y^2 - 4y}$$

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6. (*) [5 points each] All parts of this question refer to the function $f(x)$ whose *derivative* $f'(x)$ is plotted in Figure 1 above. Assume the domain of f contains only the interval shown in the plot.

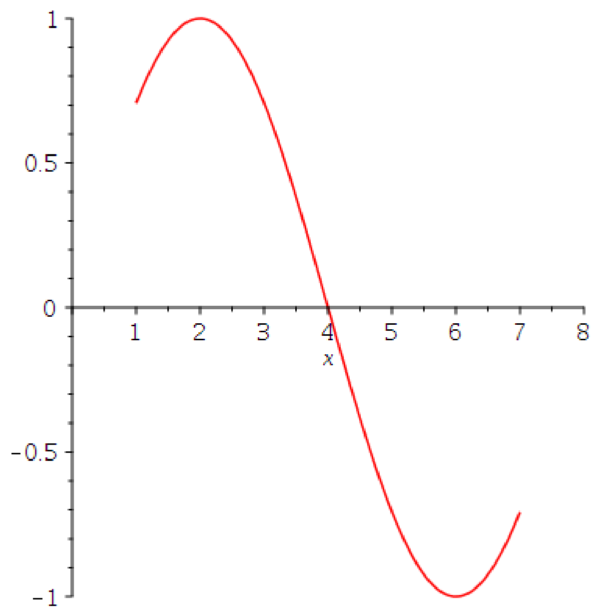


Figure 1: Plot for question 6.

(a) On which interval(s) is $f(x)$ decreasing?

Answer: On the interval $(4, 7)$.

(b) On which interval(s) is $y = f(x)$ concave up?

Answer: On $(1, 2) \cup (6, 7)$

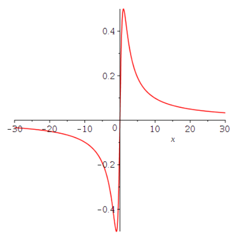
(c) How many inflection points does f have?

Answer: Two inflection points at $x = 2, 6$

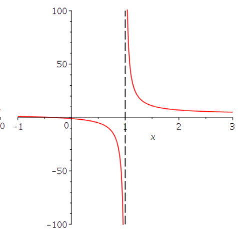
7. (*). [20 points] If the ticket price for the upcoming "Cats" movie is set at \$10, then 1000 tickets for the first showing will be sold. However, for each \$0.25 increase in the ticket price, the number of tickets sold will go down by 10 tickets. Each person who buys a ticket will also purchase \$10 of nachos and drinks at the concession stand. How should the ticket price be set to maximize the revenue earned by the theater from the ticket and concession sales?

Answer: Let x be the number of \$0.25 increases over the \$10 level. Then with the ticket price at $10 + 0.25x$, $1000 - 10x$ tickets will be sold, yielding revenue of $(1000 - 10x)(10 + 0.25x)$. In addition, $10(1000 - x)$ dollars in nachos and drinks will be sold, yielding total revenue of

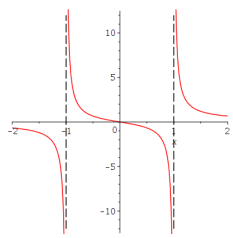
$$R(x) = (1000 - 10x)(10 + 0.25x) + 10(1000 - 10x) = 20000 + 50x - 2.5x^2$$



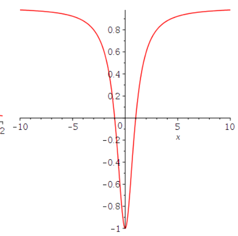
(a) Plot I



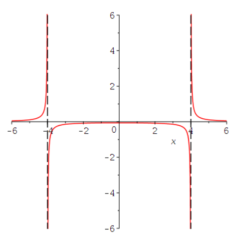
(b) Plot II



(c) Plot III



(d) Plot IV



(e) Plot V

We differentiate and set equal to zero to find critical points:

$$R'(x) = 50 - 5x = 0$$

When $x = 10$, and $R''(x) = -5 < 0$, so this is a local and global maximum for the revenue. The ticket price should be $\$(10 + 0.25 \cdot 10) = \$12.50$.

8. (*) [3 points each] By considering the location of vertical and Horizontal asymptotes, x- and y-axis intercepts, etc. determine which of the following functions matches each graph. Circle the number of the graph showing each of the following functions.

(a) $f(x) = \dfrac{x}{x^2 - 1}$

Answer: III – vertical asymptotes at $x = \pm 1$ and horizontal asymptote at $y = 0$.

(b) $f(x) = \dfrac{x}{x^2 + 1}$

Answer: I – no vertical asymptotes and horizontal asymptote at $y = 0$.

(c) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

Answer: IV – no vertical asymptotes and horizontal asymptote at $y = 1$.

(d) $f(x) = \dfrac{1}{x^2 - 16}$

Answer: V – vertical asymptotes at $x = \pm 4$.

(e) $f(x) = \dfrac{3x + 1}{x - 1}$

Answer: II – vertical asymptote at $x = 1$ (and not -1).