College of the Holy Cross, Fall 2016
Math 135, Section 1, Midterm 1 Solutions
Friday, September 23

I. Match the plots below with the following formulas. Note that there is an extra plot that does not match any of these formulas.

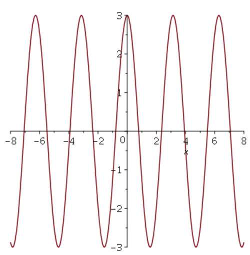
(5) A) $y = (x+1)^2 + 2$ is Plot: 4 (the $x+1$ shifts the parabola to the left, not the right)

(5) B) $y = 3\cos(2x)$ is Plot: 1 (the $2x$ makes the period equal to $\pi \doteq 3.14$, so the usual cosine graph is compressed horizontally)

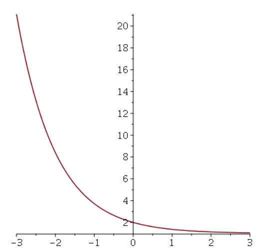
(5) C) $y = 1 + e^{-x}$ is Plot: 2 (think: $y = e^x$ reflected across the y-axis, then shifted up)

(5) D) $y = \sin(x/2)$ is Plot: 5 (the $x/2$ makes the period equal to $4\pi \doteq 12.6$, so the usual sine graph is stretched horizontally).

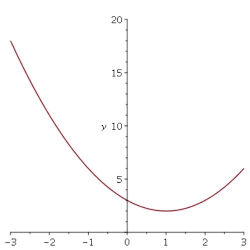
Plot 1:



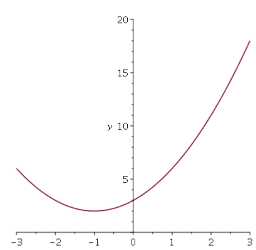
Plot 2:



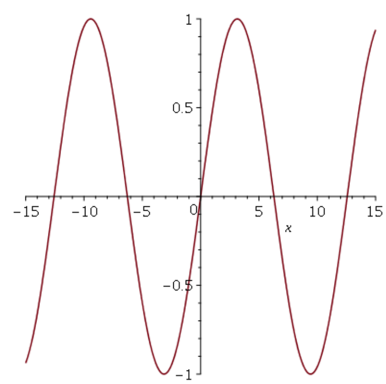
Plot 3:



Plot 4:



Plot 5:



II. The manager of a furniture factory has collected the following data for the cost of manufacturing chairs.

# Chairs (per day) C	Manufacturing Cost (in dollars) M
100	2400
150	3100
250	4500
300	5200

(10) A) Given that M is a linear function of C, determine a formula for it, using the correct labeling of the variables (that is, an answer expressed in terms of x and y will not receive full credit).

The slope is $m = \frac{3100-2400}{150-100} = 14$ so by the point slope form, we get $M - 2400 = 14(C - 100)$, or $M = 14C + 1000$.

Cost function: $\boxed{M - 2400 = 14(C - 100) \text{ or } M = 14C + 1000}$

(5) B) How much additional cost is incurred by manufacturing each additional chair?

This is $M(C + 1) - M(C) = \$14$, namely the number value of the slope with units of dollars.

(5) C) What does the M-intercept represent in terms of cost?

The M-intercept of 1000 represents the cost per day if no chairs are actually manufactured ($C = 0$). These are often called *fixed costs* – things like the maintenance costs of the factory, taxes, labor costs for employees who cannot be laid off even if no manufacturing happens, etc.

(5) D) Using your formula, determine how much it will cost to produce 350 chairs per day.

Cost: $\boxed{(14)(350) + 1000 = \$\ 5900}$

III. Given $f(x) = x^2 - 6x + 1$ and $g(x) = \sqrt{3x - 2}$, answer the following questions.

(10) A) Find the domain of $f(x)$ and the domain of $g(x)$.

The domains here are the sets of all real x that can be substituted into the formulas to yield a well-defined result. For f there are no restrictions. For g, we must have $3x - 2 \geq 0$, so $x \geq \frac{2}{3}$, or $[\frac{2}{3}, +\infty)$.

Domain of f: $\boxed{\text{all real } x, \text{ or } (-\infty, +\infty)}$

Domain of g: $\boxed{\text{all real } x \geq \frac{2}{3}, \text{ or } \left[\frac{2}{3}, +\infty\right)}$

(5) B) What is the domain of the function $g(x)/f(x)$?

Now we must be able to substitute an x that makes sense for g, and that also avoids making $f(x) = 0$. $f(x) = 0$ when $x = 3 \pm 2\sqrt{2} \doteq .17, 5.83$ (quadratic formula, or complete the square). The first of these is not in the domain of $g(x)$ so it is irrelevant. The second is, so we must leave it out:

Domain of $g(x)/f(x)$: $\boxed{\left[\frac{2}{3}, 3 + 2\sqrt{2}\right) \cup (3 + 2\sqrt{2}, +\infty), \text{ or something equivalent}}$

(5) C) Find the function $(g \circ f)(x)$. $\boxed{(g \circ f)(x) = g(f(x)) = \sqrt{3(x^2 - 6x + 1) - 2} = \sqrt{3x^2 - 18x + 1}}$

IV. Answer the following questions.

(5) A) Find all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $|\tan x| > 1$.

This is true if $\tan(x) > 1$ or $\tan(x) < -1$. The first occurs for x between $\frac{\pi}{4}$ and $\frac{\pi}{2}$; the second occurs for x between $-\frac{\pi}{2}$ and $-\frac{\pi}{4}$.

Values of x: $\boxed{\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)}$

Note: Equivalent answers like: all x with $-\frac{\pi}{2} < x < -\frac{\pi}{4}$ or $\frac{\pi}{4} < x < \frac{\pi}{2}$ are also OK.

(5) B) If $\sin\theta = \dfrac{2}{3}$ and $\dfrac{\pi}{2} < \theta < \pi$, give the exact value of $\cos\theta$.

We can use the basic trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for this: $(\frac{2}{3})^2 + \cos^2(\theta) = 1$ so $\cos^2(\theta) = \frac{5}{9}$ and $\cos(\theta) = \pm\frac{\sqrt{5}}{3}$. Since θ is between $\frac{\pi}{2}$ and π, the cosine must be negative, so the correct answer is:

$\cos\theta$: $\boxed{-\frac{\sqrt{5}}{3}}$

(5) C) Express as a single logarithm: $\dfrac{1}{2}\ln 5 - 4\ln 2 + \ln 10$. (This means your answer should be in the form ln(some exact number), not a decimal approximation.)

Use the properties of logarithms: $\ln(A)+\ln(B)=\ln(AB)$, $\ln(A)-\ln(B)=\ln(A/B)$, and $p\ln(A)=\ln(A^p)$. Then

$$\frac{1}{2}\ln 5 - 4\ln 2 + \ln 10 = \ln\left(\frac{5^{1/2}\cdot 10}{2^4}\right).$$

Single logarithm: $\boxed{\ln\left(\frac{5^{1/2}\cdot 10}{2^4}\right) = \ln\left(\frac{5\sqrt{5}}{8}\right)}$

V. Consider the function $f(x)=\frac{1}{5}e^{x+2}-3$.

(15) A) Given that f is one-to-one, find a formula for the inverse function of f.

Set up $y=\frac{1}{5}e^{x+2}-3$ and solve for x:

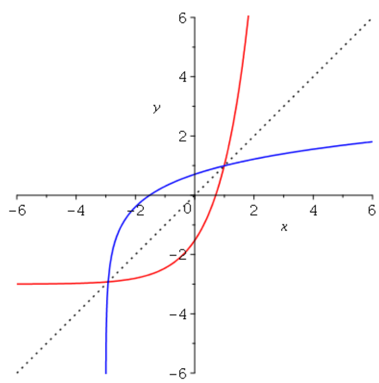
$$\begin{aligned} 5(y+3) &= e^{x+2}, \text{so after taking natural log of both sides} \\ \ln(5(y+3)) &= x+2 \\ x &= \ln(5(y+3))-2. \end{aligned}$$

We can swap the variables to write the inverse function as a function of x:

$f^{-1}(x) = \boxed{\ln(5(x+3))-2}$

(10) B) In the space below, plot the graphs of the functions f and f^{-1} on the same set of axes. Label one point on each graph with its coordinates.

Here are the graphs:



Note that $f(x) > -3$ for all x and $y=f(x)$ is approaching the horizontal line $y=-3$ as $x\to-\infty$. Because of this, the graph $y=f^{-1}(x)$ has a vertical asymptote at $x=-3$. It is obtained by reflecting $y=f(x)$ across the line $y=x$. The red curve is $y=f(x)$; it contains the point $\left(0,\frac{e^2}{5}-3\right)\doteq(0,-1.52)$. The blue curve is $y=f^{-1}(x)$, obtained by reflecting $y=f(x)$ across the line $y=x$; it contains the point $\doteq(-1.52,0)$.