**College of the Holy Cross, Fall Semester, 2016**
**MATH 135, section 1, Solutions for Midterm 3**
**Friday, December 2**

1. Find $\frac{dy}{dx}$ and simplify:

   A) (10) $y = \dfrac{x^4 + 2x}{x^3 + x + 1}$

   *Solution:* By the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^3 + x + 1)(4x^3 + 2) - (x^4 + 2x)(3x^2 + 1)}{(x^3 + x + 1)^2} \\ &= \frac{x^6 + 3x^4 + 2}{(x^3 + x + 1)^2} \end{aligned}$$

   B) (10) $y = \ln(3\sin(x) - 4\tan(x))$

   *Solution:* By the chain rule and the derivative formula for ln:

$$\frac{dy}{dx} = \frac{3\cos(x) - 4\sec^2(x)}{3\sin(x) - 4\tan(x)}.$$

   C) (10) $y = \sin^{-1}(e^{-2x})$

   *Solution:* By the chain rule and the derivative formula for $\sin^{-1}$:

$$\frac{dy}{dx} = \frac{-2e^{-2x}}{\sqrt{1 - e^{-4x}}}.$$

   (Note: $(e^{-2x})^2 = e^{-4x}$.)

   D) (10) $x^2y^4 - 4\tan^{-1}(x^2) + y = 0$ (use implicit differentiation)

   *Solution:* Using implicit differentiation,

$$4x^2y^3\frac{dy}{dx} + 2xy^4 - \frac{8x}{1 + x^4} + \frac{dy}{dx} = 0.$$

   So solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{-2xy^4 + \frac{8x}{1+x^4}}{4x^2y^3 + 1} = \frac{-2xy^4 - 2x^5y^4 + 8x}{(1 + x^4)(4x^2y^3 + 1)}$$

   (either form is OK).

2. (20) A stationary observer watches a weather balloon being launched from a point 500 feet away from her position. The balloon rises at a rate of 30 feet per second. How fast is the distance between the balloon and the observer changing when the balloon is 375 feet above the ground?

*Solution:* At all times, the observer, the balloon and the launch point are at the corners of a right triangle. Calling $y$ the height of the balloon, and $z$ the distance from the observer to the balloon, the Pythagorean theorem gives

$$z^2 = y^2 + 500^2.$$

Taking $\frac{d}{dt}$ we get

$$2z\frac{dz}{dt} = 2y\frac{dy}{dt}.$$

At the time the question is asking about, $y = 375$ and $\frac{dy}{dt} = 30$. By the Pythagorean theorem, $z = \sqrt{375^2 + 500^2} = 625$. Hence

$$\frac{dz}{dt} = \frac{375 \cdot 30}{625} = 18(\text{ft/sec}).$$

3. All parts of this question refer to the plots in Figure 1. Assume the whole domain of the functions is the interval $[-2, 8]$ shown (don't try to extrapolate what might happen on a larger interval).

   (A) (3) Is $A'(4)$ positive or negative? Answer: $A(x)$ is increasing at $x = 4$, so $A'(4)$ is *positive.*

   (B) (3) At how many different points is $B'(x) = 0$? Estimate the $x$-values from the graph. Answer: There are two such points at about $x = 0, 2.7$ (anything between 2 and 3 closer to 3 is OK).

   (C) (3) On the $x$-interval $(0, 2)$, is $A''(x)$ positive or negative? Answer: $y = A(x)$ is *concave up* on that interval so $A''(x) > 0$.

   (D) (3) On the $x$-interval $(0, 2)$ is $B'(x)$ positive or negative? Answer: $y = B(x)$ is *increasing on that interval*, so $B'(x) > 0$.

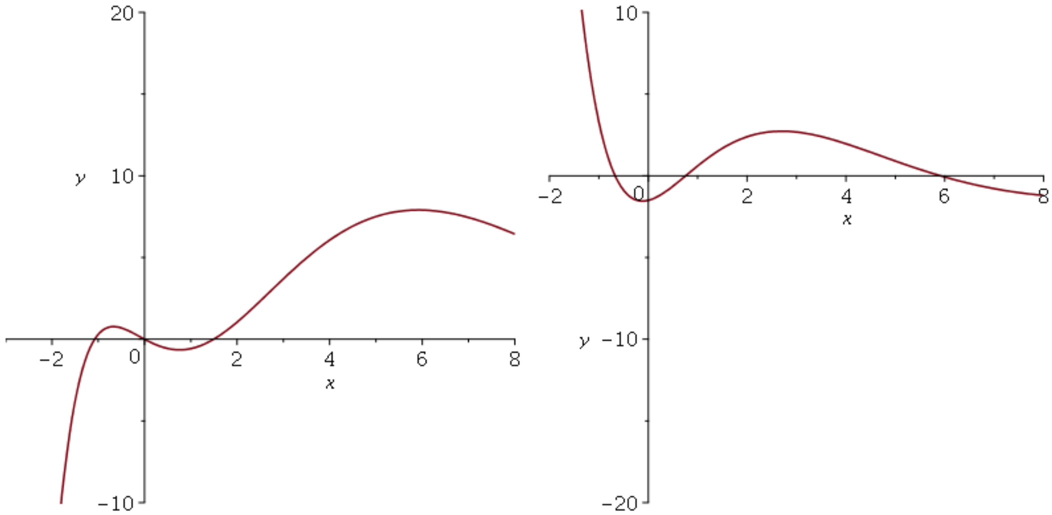
   (E) (3) One of the two functions $A(x)$ and $B(x)$ is the derivative of the other. Which is which? Answer: $B(x)$ is the derivative of $A(x)$. (Note $B(x) = 0$ when $A$ has local maximum and local minimum points – for instance at $x \doteq -0.7, +0.8, 6$.)

4. All parts of this problem refer to $f(x) = x^4 - 2x^2 - 1$.

   (A) (10) Find all critical points of $f(x)$.

   *Solution:* $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1)$. So $f(x)$ has critical points at $x = -1, 0, 1$.

   (B) (10) Find the absolute maximum and minimum values of $f(x)$ on the interval $[0, 3]$.




(a) $y = A(x)$ (b) $y = B(x)$

Figure 1: Plots for Problem 3

*Solution:* Of the $x$'s we found in part A, both $x = 0$ and $x = 1$ are in this interval. $f(0) = -1$, $f(1) = -2$ and $f(3) = 81 - 18 - 1 = 62$. The minimum is $f(1) = -2$ and the maximum is $f(3) = 62$.

(C) (5) Which of the critical points you found in part A are local maximum points and which are local minimum points? (Any correct method for this is OK.)

*Solution 1:* By the First Derivative Test, $f'(x)$ changes from negative to positive at $x = \pm 1$ so those are local minima. Then $f'(x)$ changes from positive to negative at $x = 0$, so that is a local maximum.

*Solution 2:* By the Second Derivative Test, $f''(x) = 12x^2 - 4$. We have $f''(-1) = 8 = f''(1)$. These values are positive so $f$ has local minima at $x = \pm 1$. But $f''(0) = -4 < 0$. So this is a local maximum.