MATH 135 – Calculus 1
Sample Questions for Exam 2 – Answers and Solutions
October 10, 2016

1. An object moves along a straight line path with position given by $x(t) = 4t^2 + t - 7$, (t in seconds, x in feet).

   (a) What is the average velocity of the object over the interval $[0, 5]$ of t-values?

   Solution: The average velocity is:

   $$v_{ave} = \frac{x(5) - x(0)}{5 - 0} = \frac{98 - (-7)}{5} = 21 \text{ ft/sec}$$

   (b) Fill in the following table with average velocities computed over the indicated intervals. Using this information, estimate the *instantaneous velocity* at $t = 2$.

   Solution:

interval	$[2, 2.5]$	$[2, 2.05]$	$[2, 2.005]$	$[2, 2.0005]$
ave.vel.	19.0	17.2	17.02	17.002

   It looks as though the average velocity is tending to 17 as the length of the interval goes to 0.

   (c) Construct a similar table for intervals *ending* at $t = 2$ and repeat the calculations in the previous part. If you estimate the instantaneous velocity at $t = 2$ using this new information, does your result agree with what you did before (it should!)

   Solution:

interval	$[1.5, 2]$	$[1.95, 2]$	$[1.995, 2]$	$[1.9995, 2]$
ave.vel.	15.0	16.8	16.98	16.998

   It looks as though the average velocity is tending to 17 as the length of the interval goes to 0 again.

2. (a) What is the slope of the secant line to the graph $y = x^3 + 1$ through the points with $x = 1$ and $x = 2$?

   Solution: The slope of the secant line is

   $$m_{sec} = \frac{(2^3 + 1) - (1^3 + 1)}{2 - 1} = \frac{9 - 2}{1} = 7.$$

   (b) What is the slope of the secant line to the graph $y = x^3 + 1$ through the points with $x = 1$ and $x = 1 + h$ for a general h?

   Solution: The slope of the secant line is

   $$m_{sec} = \frac{(1 + h)^3 + 1 - (1^3 + 1)}{1 + h - 1} = \frac{3h + 3h^2 + h^3}{h}.$$

(c) The slope of the tangent line to $y = x^3 + 1$ at $x = 1$ would be obtained from what limit?

Solution: The slope of the tangent is

$$m_{tan} = \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h}.$$

(d) Estimate the limit in the previous part numerically (as in the first question).

Solution:

h	1	.5	.05	.005
$(3h + 3h^2 + h^3)/h$	7	4.75	3.1525	3.015025

It seems the limit as $h \to 0$ is about 3.

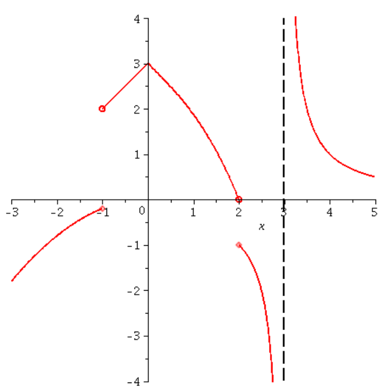
(e) Evaluate the limit exactly using our algebraic techniques.

Solution:

$$\lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(3 + 3h + h^2)}{h} = \lim_{h \to 0} 3 + 3h + h^2 = 3$$

(cancelling the h's top and bottom, then using the continuity of $3 + 3h + h^2$ as a function of h).

3. The graph of a function f is shown below with several points marked. Find all the marked points at which the following are true, and give explanations for your answers.



(a) f has an infinite discontinuity – *Answer:* $x = 3$.

(b) f has jump discontinuity – *Answer:* $x = -1$ and $x = 2$. We have

$$\lim_{x \to -1^-} f(x) \doteq -0.3, \text{ and } \lim_{x \to -1^+} f(x) = 2$$

and

$$\lim_{x \to 2^-} f(x) = 0, \text{ and } \lim_{x \to 2^+} f(x) = -1.$$

(c) f has a removable discontinuity – *Answer:* There are no removable discontinuities for the function given here. A removable discontinuity would be an $x = b$ where $\lim_{x\to b} f(x) = L$ exists ("from both sides") but $L \neq f(b)$.

(d) f is continuous – *Answer:* f is continuous at all points shown except $x = -1, 2, 3$.

4. Compute the indicated limits. Show all work for full credit.

(a) $\lim_{x\to 1} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4}$

Solution: This is not an indeterminate form, and the denominator is not zero at 1, so the answer can be found by continuity:

$$\lim_{x\to 1} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4} = \frac{-4}{1} = -4.$$

(b) $\lim_{x\to 2} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4}$

Answer: This is a 0/0 form. To try to evaluate, we aim to factor the bottom and cancel factors of $x - 2$:

$$\lim_{x\to 2} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4} = \lim_{x\to 2} \frac{(x-2)(3x+1)}{(x-2)(x-2)} = \lim_{x\to 2} \frac{3x+1}{x-2}.$$

This is not indeterminate any more, but the bottom is still 0 at $x = 2$. So this limit does not exist. (The function has an infinite discontinuity at $x = 2$.)

(c) $\lim_{x\to\infty} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4}$

Solution: For this one, we multiply the top and bottom by $\frac{1}{x^2}$:

$$\begin{aligned} lim_{x\to\infty} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4} &= \lim_{x\to\infty} \frac{(3x^2 - 5x - 2)\frac{1}{x^2}}{(x^2 - 4x + 4)\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{3 - \frac{5}{x} - \frac{2}{x^2}}{1 - \frac{4}{x} + \frac{4}{x^2}} \\ &= \frac{3 - 0 - 0}{1 - 0 + 0} = 3. \end{aligned}$$

This says that the graph of this function has a horizontal asymptote at the line $y = 3$.

(d) $\lim_{x\to 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x - 2}$

Solution:

$$\begin{aligned}\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} &= \lim_{x\to 2}\frac{4-x^2}{4x^2(x-2)}\\ &= \lim_{x\to 2}\frac{(2-x)(2+x)}{4x^2(x-2)}\\ &= \lim_{x\to 2}\frac{-(2+x)}{4x^2} = \frac{-1}{4}.\end{aligned}$$

(e) $\lim_{t\to 0}\dfrac{\sin(6t)}{\sin(7t)}$

Solution: For this one, we need to use the formula $\lim_{u\to 0}\frac{\sin(u)}{u}=1$ from Section 2.6. We have

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(6t)}{\sin(7t)} &= \lim_{t\to 0}\frac{6\cdot\sin(6t)/(6t)}{7\cdot\sin(7t)/(7t)}\\ &= 6/7.\end{aligned}$$

(f) $\lim_{h\to 0}\dfrac{\sqrt{h+9}-\sqrt{9}}{h}$

Solution: Multiply top and bottom by the conjugate radical:

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{h+9}-\sqrt{9}}{h} &= \lim_{h\to 0}\frac{(\sqrt{h+9}-\sqrt{9})(\sqrt{h+9}+\sqrt{9})}{h(\sqrt{h+9}+\sqrt{9})}\\ &= \lim_{h\to 0}\frac{(h+9)-9}{h(\sqrt{h+9}+\sqrt{9})}\\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{h+9}+\sqrt{9})}\\ &= \lim_{h\to 0}\frac{1}{(\sqrt{h+9}+\sqrt{9})}\\ &= \frac{1}{2\sqrt{9}} = \frac{1}{6}.\end{aligned}$$

5. Suppose you know each of the following conditions. What can you say about $\lim_{x\to c} f(x)$ for the indicated c? Why?

   (a) $x^2+x \le f(x) \le x^3+3$ for all real x, at $c=0$.

   Solution: $f(x)$ is not "squeezed" in this case because $l(x)=x^2+x$ has the value $l(0)=0$, which is strictly less than $u(0)=3$ for $u(x)=x^3+3$. The limit of $f(x)$ as $x\to 0$ could be any number between 0 and 3, or it might not exist at all.

(b) $-x^2 + 2x \le f(x) \le x^4 - 4x^3 + 6x^2 - 4x + 2$ for all real x, at $c = 1$

Solution: Now we have $l(1) = 1$ and $u(1) = 1$. So $f(x)$ is "squeezed" and the Squeeze Theorem says $\lim_{x\to 1} f(x) = 1$ also.

(c) $f(x) = x \sin\left(\frac{1}{x}\right)$ for all real $x \ne 0$, at $c = 0$.

Solution: This one is trickier because we are not given $l(x)$ and $u(x)$. However, note that for all $x \ne 0$, $-1 \le \sin\left(\frac{1}{x}\right) \le 1$. Therefore, $-x \le f(x) \le x$ for all $x \ne 0$. Hence the Squeeze Theorem does apply and we can see $\lim_{x\to 0} f(x) = 0$. (This $f(x)$ has a graph that looks like the picture from Exercise 56 B on page 71.)

6. (a) What is the definition of the derivative of a function f at $x = a$ in its domain?

Solution: $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$, if the limit exists. The limit $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ also computes $f'(a)$. These can be seen to be equivalent if you let $x - a = h$ in the second form.

(a) Using the definition (not the shortcut rules), find $f'(x)$ for $f(x) = 3x^3 - 2x^2 + 1$ at a general $x = a$.

(b) *Solution:*

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{(3(a+h)^3 - 2(a+h)^2 + 1 - (3a^3 - 2a^2 + 1)}{h} \\
&= \lim_{h\to 0} \frac{3a^3 + 9a^2h + 9ah^2 + 3h^3 - 2a^2 - 4ah - 2h^2 + 1 - 3a^3 + 2a^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{(9a^2 - 4a + 9ah - 2h + 3h^2)h}{h} \\
&= \lim_{h\to 0} 9a^2 - 4a + 9ah + 2h + 3h^2 \\
&= 9a^2 - 4a.
\end{aligned}
$$

(c) Using the definition (not the shortcut rules), find $f'(x)$ for $f(x) = x^{1/2}$ at a general $x = a > 0$.

Solution:

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}} \\
&= \lim_{h\to 0} \frac{a + h - a}{h \cdot (\sqrt{a+h} + \sqrt{a})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} \\
&= \frac{1}{2\sqrt{a}}.
\end{aligned}
$$

(b) Using the definition (not the shortcut rules), find $f'(x)$ for $f(x) = \frac{1}{x^2}$ at a general $x = a \neq 0$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h} \\ &= \lim_{h\to 0} \frac{a^2 - (a+h)^2}{ha^2(a+h)^2} \\ &= \lim_{h\to 0} \frac{-2ah - h^2}{ha^2(a+h)^2} \\ &= \lim_{h\to 0} \frac{-2a - h}{a^2(a+h)^2} \\ &= \frac{-2a}{a^4} = \frac{-2}{a^3}. \end{aligned}$$

7. (a) State and prove the product rule for derivatives.

(b) *Solution:* The rule says that if f and g are both differentiable at x, then $f(x)g(x)$ is also differentiable at x and

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

The reason this is true is that

$$\begin{aligned} \frac{d}{dx}(f(x)g(x)) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x)f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x)f(x)g(x)}{h} \\ &= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

Along the way here, we used the fact that if $f'(x)$ exists, then $\lim_{h\to 0} f(x+h) = f(x)$ (since otherwise the difference quotient for f would not be an indeterminate form as $h \to 0$ and the limit could not exist).

(c) Use the product rule to find $f'(x)$ for $f(x) = (4x^3 - 12x^2 + 1)e^x$.

(d) *Solution:*

$$f'(x) = (4x^3 - 12x^2 + 1)e^x + e^x(12x^2 - 24x) = (4x^3 - 24x + 1)e^x$$

(e) Find $f'(x)$ for

$$f(x) = \frac{x^2 - 4x + 1}{x^3 + 2}$$

(f) *Solution:* By the quotient rule:

$$f'(x) = \frac{(x^3+1)(2x-4)-(x^2-4x+1)(3x^2)}{(x^3+2)^2} = \frac{-x^4-8x^3-3x^2+4x+8}{(x^3+2)^2}$$

(g) Find $f'(x)$ two ways: One, using the quotient rule, one *without* using the quotient rule. Verify that the result is the same in both cases. Which is easier?

$$f(x) = \frac{x^6-3x^3+x}{x^{1/2}}$$

(h) *Solution:* Using the quotient rule:

$$f'(x) = \frac{x^{1/2}(6x^5-9x^2)-(x^6-3x^3+x)\cdot\frac{1}{2}x^{-1/2}}{(x^{1/2})^2}$$

Another way to do this is to divide $x^{1/2}$ into each term on the top, and then differentiate:

$$f(x) = x^{11/2} - 3x^{5/2} + x^{1/2},$$

so then

$$f'(x) = \frac{11}{2}x^{9/2} - \frac{15}{2}x^{3/2} + \frac{1}{2}x^{-1/2}.$$

It is a "nice exercise" to show that these define the same function(!)