College of the Holy Cross, Fall 2016
Math 136, section 2, Midterm Exam 1
Friday, September 23

I.

A. (10) The following limit of a sum would equal the definite integral $\int_a^b f(x)\,dx$ for some function $f(x)$ on some interval $[a,b]$. What function and what interval?

$$\lim_{N\to\infty} \sum_{j=1}^{N} \left(37 - 3\left(1 + \frac{2j}{N}\right)^2\right) \cdot \frac{2}{N}.$$

Solution: This is the limit of the R_N Riemann sum for $f(x) = 37 - 3x^2$ on the interval $[1,3]$, so the limit will equal $\int_1^3 37 - 3x^2\,dx$. (Note, other answers would be equally correct, such as $f(x) = 37 - 3(1+x)^2$ on $[0,2]$.)

B. (5) For any given N (i.e. without taking the limit), is the sum in part A greater than or less than the integral? Explain how you can tell.

Solution: This is a right-hand sum for an *decreasing* function ($f(x) = 37 - 3x^2$ has $f'(x) = -6x < 0$ for all $x > 0$, and $37 - 3x^2 > 0$ for $x \in [1,3]$). Hence the sum must be *less* than the value of the integral.

C. (5) Would you expect the Midpoint Riemann sum or the sum in part A (i.e. before taking the limit as $N \to \infty$) to be closer to the value of your integral? Explain.

Solution: The Midpoint Riemann sum is certainly closer. Since f is decreasing, each of the rectangles whose areas we add to get the Midpoint Riemann sum misses some area to the left of the midpoint, but has some extra area to the right of the midpoint. These areas would tend to balance each other out and the area of the midpoint rectangle would be close to the exact area under the graph on that subinterval, at least if the Δx was sufficiently small (equivalently, N was sufficiently large).

II. All parts of this problem refer to $f(x) = x^2 + 1$ on the interval $[a,b] = [0,2]$.

A. (10) Evaluate the R_4 Riemann sum for f on this interval.

Solution: With $N = 4$, we have $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. The end points of the intervals are $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, and $x_4 = 2$. The right-hand Riemann sum equals:

$$f(1/2)\Delta x + f(1)\Delta x + f(3/2)\Delta x + f(2)\Delta x,$$

which equals

$$[((1/2)^2 + 1) + (1^2 + 1) + ((3/2)^2 + 1) + (2^2 + 1)](1/2) = 23/4 = 5.75$$

B. (15) Give a formula for R_N not involving a summation, and use it to determine $\int_0^2 x^2 + 1\ dx$.

Solution: We have $\Delta x = \frac{2}{N}$ and $x_j = \frac{2j}{N}$ for all $j = 1, \ldots, N$, so

$$\begin{aligned} R_N &= \sum_{j=1}^{N} \left(\left(\frac{2j}{N} \right)^2 + 1 \right) \cdot \frac{2}{N} \\ &= \frac{8}{N^3} \sum_{j=1}^{N} j^2 + \frac{2}{N} \sum_{j=1}^{N} 1 \\ &= \frac{8}{N^3} \left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right) + 2 \quad \text{(from power sum formula (2))} \\ &= \frac{14}{3} + \frac{4}{N} + \frac{4}{3N^2}. \end{aligned}$$

Hence

$$\int_0^2 x^2 + 1\ dx = \lim_{N \to \infty} R_N = \lim_{N \to \infty} \left(\frac{14}{3} + \frac{4}{N} + \frac{4}{3N^2} \right) = \frac{14}{3}.$$

C. (5) Use Part II of the Fundamental Theorem of Calculus to check your answer from part B.

Solution: The exact value is

$$\int_0^2 x^2 + 1\ dx = \left. \frac{x^3}{3} + x \right|_0^2 = \frac{8}{3} + 2 - 0 = \frac{14}{3}.$$

Note that this equals the limit of R_N from part B as $N \to \infty$

III. All parts of this problem refer to $f(x) = (x-2)^2(x-4)$ and

$$A(x) = \int_0^x f(t)\ dt$$

for this f.

A. (10) Where does $A(x)$ have critical points? Is each of them a local maximum, an local minimum, or neither?

Solution: By the Fundamental Theorem of Calculus, part I,

$$A'(x) = f(x) = (x-2)^2(x-4),$$

which exists for all real x and equals 0 when $x = 2, 4$. Those are the critical points of $A(x)$. As seen in the plot at the top of the next page, $f(x)$ does not change sign at $x = 2$ (it's negative on both sides). Therefore, $x = 2$ is neither a local maximum nor a local minimum. But $f(x)$ does change sign from negative to positive at $x = 4$. Therefore $x = 4$ is a local minimum of $A(x)$.

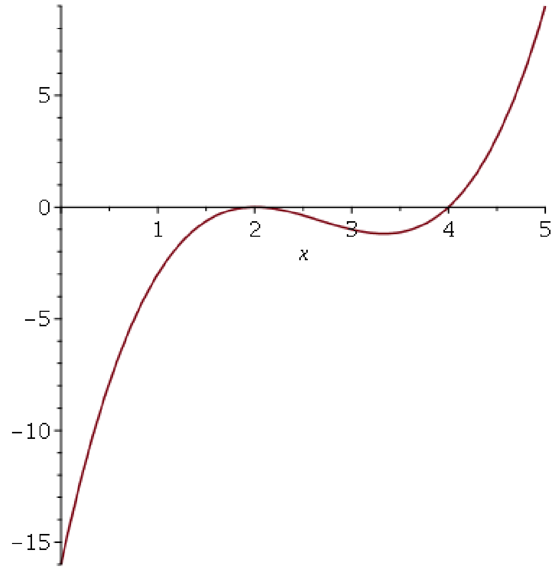

Figure 1: $y = (x-2)^2(x-4)$ on $[0,5]$

B. (5) Is $A(4)$ a positive or negative number? How can you tell? (It's not necessary to compute the value to tell – why not?)

Solution: What we said in part A shows $f(x) \le 0$ for all $x \in [0,4]$ and $f(x) < 0$ for all $x \ne 2, 4$ in that interval. Therefore $A(4) = \int_0^4 f(x)\,dx < 0$ by the interpretation of the definite integral as a signed area.

C. (5) If $B(x) = \int_1^x f(t)\,dt$, how are $A(x)$ and $B(x)$ related to each other?

Solution: By the interval union property for definite integrals, $A(x) = \int_0^1 f(t)\,dt + B(x)$, and $\int_0^1 f(t)\,dt$ is a constant (it $= \frac{-101}{12}$). (Also correct: $A'(x) = B'(x) = f(x)$ by the first part of the Fundamental Theorem of Calculus, etc.)

IV.

A. (10) Integrate with a suitable u-substitution: $\displaystyle\int_0^1 (4x^3+1)^{3/5} x^2\,dx$.

Solution: Let $u = 4x^3 + 1$. Then $du = 12x^2\,dx$, so $\frac{1}{12}du = x^2\,dx$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 5$. The definite integral goes over to

$$\frac{1}{12}\int_1^5 u^{3/5}\,du = \frac{1}{12}\cdot\frac{5}{8}\,u^{8/5}\Big|_1^5 = \frac{5}{96}[5^{8/5} - 1] \doteq .6319$$

B. (10) Integrate with a suitable u-substitution: $\displaystyle\int \frac{x\sin(3x^2)}{\cos(3x^2)+1}\,dx$.

Solution: Let $u = \cos(3x^2) + 1$. Then $du = -6x \sin(3x^2)\ dx$. So $x \sin(3x^2)\ dx = \frac{-1}{6}\ du$ and the integral goes over to

$$\frac{-1}{6} \int \frac{du}{u} = \frac{-1}{6} \ln |u| + C = \frac{-1}{6} \ln | \cos(3x^2) + 1| + C.$$

C. (10) Integrate with any applicable method we have discussed: $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2+1}} dx$

Solution: The $x\ dx$ in the numerator is, up to a constant, the derivative of $u = x^2 + 1$ in the radical. The integral

$$= \frac{1}{2} \int_{u=1}^{u=2} u^{-1/2}\ du = u^{1/2}\Big|_1^2 = \sqrt{2} - 1.$$