College of the Holy Cross, Fall 2016
Math 136 – Solutions for Midterm Exam 3
December 1

I. [20 points] Integrate with the partial fraction method: $\displaystyle\int \frac{2x^2+3}{x^3+9x}\,dx$

Solution: The denominator factors as $x^3 + 9x = x(x^2 + 9)$, where $x^2 + 9 = 0$ has no real roots. Therefore, the partial fractions will have the form:

$$\frac{2x^2+3}{x^3+9x} = \frac{A}{x} + \frac{Bx+C}{x^2+9},$$

so

$$2x^2 + 3 = A(x^2+9) + (Bx+C)x = (A+B)x^2 + Cx + 9A.$$

Equating coefficients of like powers of x, we get $9A = 3$, so $A = \frac{1}{3}$, $C = 0$, and $A + B = 2$, so $B = \frac{5}{3}$. Then we integrate:

$$\begin{aligned}\int \frac{2x^2+3}{x^3+9x}\,dx &= \frac{1}{3}\int \frac{1}{x}\,dx + \frac{5}{3}\int \frac{x}{x^2+9}\\ &= \frac{1}{3}\ln|x| + \frac{5}{6}\ln|x^2+9| + C.\end{aligned}$$

II. For each integral, say why the integral is *improper*, then set up and evaluate the appropriate limits to determine whether the integral converges. If so, find its value; if not, say "does not converge." (Full credit will be given only for the correct limit calculation.)

A. [15 points] $\displaystyle\int_2^\infty xe^{-x}\,dx.$

Solution: This is improper because of the infinite upper limit of integration. To determine whether the integral converges, we compute

$$\lim_{b\to\infty}\int_2^b xe^{-x}\,dx,$$

which is done by parts: $u = x$, so $du = dx$ and $dv = e^{-x}\,dx$, so $v = -e^{-x}$:

$$\begin{aligned} &= \lim_{b\to\infty}\left(-xe^{-x}\Big|_2^b + \int_2^b e^{-x}\,dx\right)\\ &= \lim_{b\to\infty}\left(-\frac{b}{e^b} + \frac{2}{e^2} - e^b + \frac{1}{e^2}\right)\\ &= \frac{3}{2e},\end{aligned}$$

(because $\lim_{b\to\infty}\frac{b}{e^b} = \lim_{b\to\infty}\frac{1}{e^b} = 0$ by L'Hopital's Rule). The integral *converges* to $\frac{3}{2e}$.

B. [15 points] $\displaystyle\int_0^3 \frac{1}{x^2+4x}\,dx$

Solution: This is improper because $\frac{1}{x^2+4x}$ has a discontinuity (a vertical asymptote) at $x = 0$, which is the left hand end-point of the interval $[0, 3]$. It also has a second vertical asyptote at $x = -4$, but that is not relevant for this question. To determine whether the integral converges we need to find

$$\lim_{a\to 0^+} \int_a^3 \frac{1}{x^2+4x}\,dx.$$

This integral is one we can do with partial fractions:

$$\frac{1}{x^2+4x} = \frac{A}{x} + \frac{B}{x+4}$$

which yields $1 = A(x+4) + Bx$, so $A + B = 0$ and $4A = 1$. Therefore $A = \frac{1}{4}$ and $B = -\frac{1}{4}$. As a result, we have

$$\begin{aligned}
\lim_{a\to 0^+} \int_a^3 \frac{1}{x^2+4x}\,dx &= \lim_{a\to 0^+} \int_a^3 \frac{1}{4x} - \frac{1}{4(x+4)}\,dx \\
&= \lim_{a\to 0^+} \left(\frac{1}{4}\ln|x| - \frac{1}{4}\ln|x+4| \Big|_a^3 \right) \\
&= \lim_{a\to 0^+} \frac{1}{4}\left(\ln(3) - \ln(a) - \ln(7) + \ln(a+4)\right)
\end{aligned}$$

This does not exist because $\ln_{a\to 0^+} \ln(a)$ does not exist (the natural log has a vertical asymptote at 0 from the right). Therefore the integral *does not exist* (or diverges).

III. Both parts of this problem deal with the differential equation $\frac{dy}{dx} = \frac{y}{\sqrt{x}}$.

A. [15 points] Find the general solution $y(x)$ of the equation by separating variables and integrating.

Solution: We have

$$\int \frac{1}{y}\,dy = \int x^{-1/2}\,dx$$

so

$$\ln(y) = 2x^{1/2} + C$$

and hence

$$y = De^{2x^{1/2}}$$

where $D = \pm e^C$ is an arbitrary constant.

B. [5 points] Find the particular solution $y(x)$ satisfying the initial condition $y(1) = 4$ and compute the exact value of $y(2)$.

Solution: The equation $y(1) = 4$ says $4 = De^2$, so $D = 4e^{-2}$. Then $y(2) = 4e^{-2}e^{2\sqrt{2}} = 4e^{2\sqrt{2}-2}$.

IV. [10 points] A population y satisfies the logistic growth equation

$$\frac{dy}{dt} = (.3)y\left(1 - \frac{y}{100}\right)$$

and the initial condition $y(0) = 3$. What is the population level when the population is *growing the fastest*?

Solution 1: $\frac{dy}{dt}$ is the rate of change of the population. So we are looking for y that maximizes

$$(.3)y\left(1 - \frac{y}{100}\right) = (.3)\left(y - \frac{y^2}{100}\right).$$

The graph of this function is a parabola opening down. The maximum occurs where the derivative of this quadratic function of y is zero:

$$(.3)\left(1 - \frac{y}{50}\right) = 0,$$

so $y = 50$.

Solution 2: The general solution of this logistic equation is

$$y = \frac{100}{1 + de^{-.3t}}.$$

With the initial condition $y(0) = 3$, we have $3 = \frac{100}{1+d}$, so $d = 97/3$. Then the rate of change of y as a function of t is given by:

$$\frac{dy}{dt} = \frac{-970e^{-.3t}}{(1 + \frac{97}{3}e^{-.3t})^2}.$$

To determine when this is a maximum, we can differentiate again with the quotient rule. After simplification:

$$\frac{d^2y}{dt^2} = \frac{(25043e^{-0.3t} - 7857)e^{-0.3t}}{(1 + \frac{97}{3}e^{-.3t})^3}$$

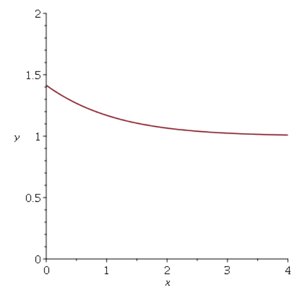
This is zero when

$$25043e^{-0.3t} - 7857 = 0, \quad \text{or} \quad t \doteq 11.59.$$

Then $y(11.59) \doteq 50$.

V.

A. [10 points] A function $y = f(x)$ is plotted below.



Check the appropriate boxes for statements about $\int_0^4 f(x)\,dx$:

Solution: The graph is concave up, which shows that the midpoint approximation is an *underestimate* and the trapezoidal rule approximation is an *overestimate*.

B. [10 points] Suppose you know that the function $f(x)$ plotted here is $g''(x)$ for some other function $g(x)$. How big would you need to take N to get a midpoint rule approximation to $\int_0^4 g(x)\,dx$ with error $< 10^{-5}$?

Solution: From the error bound for the midpoint rule, we have

$$|\text{Error}(M_N)| \leq \frac{K_2 \cdot (4-0)^3}{24N^2},$$

where $K_2 = \max_{x\in[0,3]} |g''(x)|$. From the graph we can take $K_2 = 1.5$ and then we want

$$\frac{96}{24N^2} < 10^{-5} = .00001.$$

This is equivalent to

$$N^2 > \frac{96}{24 \cdot 0.00001},$$

or

$$N > \sqrt{\frac{96}{24 \cdot 0.00001}} = 2000.$$