College of the Holy Cross, Fall 2016
Math 136, section 2 – Solutions for Midterm Exam 2
Friday, October 28

I.

A. (15) Integrate by parts: $\int x^2 \sin(5x)\ dx$

Solution: We integrate by parts twice, each time letting $u =$ power of x:

$$\begin{aligned}\int x^2 \sin(5x)\ dx &= \frac{-x^2\cos(5x)}{5} + \frac{2}{5}\int x\cos(5x)\ dx \\ &= \frac{-x^2\cos(5x)}{5} + \frac{2}{5}\left(\frac{x\sin(5x)}{5} - \frac{1}{5}\int \sin(5x)\ dx\right) \\ &= \frac{-x^2\cos(5x)}{5} + \frac{2x\sin(5x)}{25} + \frac{2\cos(5x)}{125} + C.\end{aligned}$$

B. (10) Integrate using appropriate trigonometric identities: $\int \sec^4(4x)\tan^2(4x)\ dx$.

Solution: With the even power of sec we can proceed like this:

$$= \int \sec^2(4x)(1+\tan^2(4x))\tan^2(4x)\ dx = \int \tan^2(4x)\sec^2(4x)\ dx + \int \tan^4(4x)\sec^2(4x)\ dx.$$

Each of these integrals can be handled with the substitution $u = \tan(4x)$, $du = 4\sec^2(4x)\ dx$. The integral is

$$\frac{1}{12}\tan^3(4x) + \frac{1}{20}\tan^5(4x) + C.$$

C. (20) Integrate with a trigonometric substitution: $\int \frac{x^3}{\sqrt{81 - x^2}}\ dx$. (Partial credit points will be given as follows: 5 points for the substitution equation $x = \ldots$, and the computation of dx, 5 points for the conversion to the trigonometric integral, 5 points for the integration of the trigonometric integral, 5 points for conversion back to the equivalent function of x.)

Solution: Let $x = 9\sin\theta$, so $dx = 9\cos\theta\ d\theta$. The integral goes over to

$$\int \frac{729\sin^3\theta}{9\cos\theta}\cdot 9\cos\theta\ d\theta = 729\int \sin^3\theta d\theta.$$

Since we have the odd power of sin, we can split off one power, convert the other powers to cosines, then integrate with a u-substitution:

$$= 729\int (1 - \cos^2\theta)\sin\theta\ d\theta = 729\int \sin\theta\ d\theta - 729\int \cos^2\theta\sin\theta\ d\theta.$$

In the second, we let $u = \cos\theta$ and recognize the form as $\int u^2\, du$. So the trig integral equals

$$-729\cos\theta + 243\cos^3\theta + C$$

Then from the triangle corresponding to the original $x = 9\sin\theta$, we have $\cos\theta = \frac{\sqrt{81-x^2}}{9}$ and the integral expressed in terms of x is

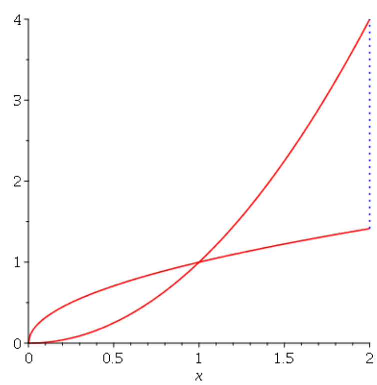
$$-81\sqrt{81-x^2} + \frac{(81-x^2)^{3/2}}{3} + C$$

D. (10) Integrate with any applicable method we have discussed: $\displaystyle\int x^4 \ln(x)\, dx$

Solution: Use integration by parts, but with $u = \ln(x)$ and $dv = x^4\, dx$. We get

$$\int x^4 \ln(x)\, dx = \frac{x^5\ln(x)}{5} - \int \frac{1}{x}\cdot\frac{x^5}{5}\, dx = \frac{x^5\ln(x)}{5} - \frac{x^5}{25} + C.$$

II. All parts of this question refer to the region R bounded by the graphs $y = \sqrt{x}$ and $y = x^2$, $x = 0$ and $x = 2$:



A. (10) Set up and compute integral(s) to find the area of R.

Solution: Between $x = 0$ and $x = 1$, the graph $y = \sqrt{x}$ lies above $y = x^2$, but this switches between $x = 1$ and $x = 2$. So the area is computed by

$$\int_0^1 \sqrt{x} - x^2\, dx + \int_1^2 x^2 - \sqrt{x}\, dx = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3\Big|_0^1 + \frac{1}{3}x^3 - \frac{2}{3}x^{3/2}\Big|_1^2 = \frac{10-4\sqrt{2}}{3} \doteq 1.45.$$

B. (10) The portion of the region R between $x = 0$ and $x = 1$ is rotated about the x-axis to generate a solid. Set up and compute an integral to find its volume.

Solution: The cross-sections by planes perpendicular to the x-axis are washers with inner radius $r_{in} = x^2$ and outer radius $r_{out} = \sqrt{x}$. So the volume is

$$\int_0^1 \pi(\sqrt{x})^2 - \pi(x^2)^2\, dx = \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\Big|_0^1\right) = \frac{3\pi}{10} \doteq .94$$

C. (10) A solid has the portion of the region R between $x = 1$ and $x = 2$ as base. The cross-sections by planes perpendicular to the x-axis are isosceles right triangles with hypotenuse extending from the lower boundary to the upper boundary of the region. Set up and compute an integral to find the volume.

Solution: An isosceles right triangle with hypotenuse h has legs of length $\frac{h}{\sqrt{2}}$ and area

$$\frac{1}{2} \cdot \frac{h}{\sqrt{2}} \cdot \frac{h}{\sqrt{2}} = \frac{h^2}{4}.$$

Here the cross-section for a general $1 \le x \le 2$ has $h = x^2 - \sqrt{x}$, so our volume is computed by

$$V = \int_1^2 \frac{(x^2 - \sqrt{x})^2}{4}\ dx$$

The Extra Credit computation – Continuing from the integral above,

$$= \frac{1}{4} \int_1^2 x^4 - 2x^{5/2} + x\ dx = \frac{1}{4} \left(\frac{x^5}{5} - \frac{4}{7} x^{7/2} + \frac{x^2}{2} \bigg|_1^2 \right) = \frac{579}{280} - \frac{8\sqrt{2}}{7}.$$

III. (15) Suppose that a region R defined by $0 \le y \le f(x)$ and $a \le x \le b$ has area A and lies above the x-axis. When R is rotated about the x-axis it sweeps out a solid with volume V_1. When R is rotated about the line $y = -k$, where $k > 0$, it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, A.

Solution: Since $-k < 0$, the cross-sections of the new solid are washers with outer radius $f(x) + k$ and inner radius k. The volume V_2 equals

$$V_2 = \int_a^b \pi(f(x) + k)^2 - \pi k^2\ dx = \int_a^b \pi(f(x))^2\ dx + 2k\pi \int_a^b f(x)\ dx$$

The first integral here computes the volume V_1 when R is rotated about the x-axis and the second computes the area of the region R. Hence this equals $V_1 + 2k\pi A$. We have

$$V_2 = V_1 + 2k\pi A.$$