1. Compute the indicated limits. Show all work for full credit.

(a) (5) $\lim_{x \to 1} \frac{x^2 - 6x + 8}{2x^2 - 2x - 4}$

Solution: This is a rational function, so it is continuous at every x where the denominator is not zero. Since $2 \cdot (1)^2 - 2 \cdot 1 - 4 = -4 \neq 0$ that is true here. The limit is the same as the value of the function at 1: $= \frac{3}{-4} = -\frac{3}{4}$. Note: the same conclusion can be reached by using the limit sum, product, and quotient rules.

(b) (5) $\lim_{x \to 2} \frac{x^2 - 6x + 8}{2x^2 - 2x - 4}$

Solution: This is a 0/0 indeterminate form. We factor the top and bottom and cancel to obtain

$$\frac{x^2 - 6x + 8}{2x^2 - 2x - 4} = \frac{(x-2)(x-4)}{(x-2)(2x+2)} = \frac{x-4}{2x+2}$$

for all $x \neq 2$. This function is continuous at $x = 2$, and the limit is $\frac{-2}{6} = \frac{-1}{3}$.

(c) (5) $\lim_{x \to \infty} \frac{x^2 - 6x - 8}{2x^2 - 2x - 4}$

Solution: The limit is $\frac{1}{2}$, as can be seen by this calculation (multiply top and bottom by $\frac{1}{x^2}$):

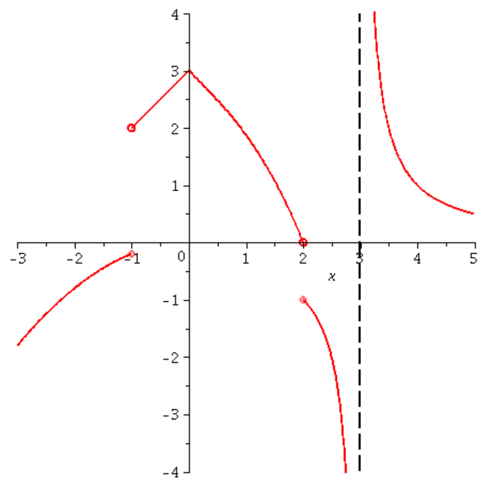
$$\begin{aligned} \lim_{x \to \infty} \frac{x^2 - 6x - 8}{2x^2 - 2x - 4} &= \lim_{x \to \infty} \frac{1 - \frac{6}{x} - \frac{8}{x^2}}{2 - \frac{2}{x} - \frac{4}{x^2}} \\ &= \frac{1}{2}. \end{aligned}$$

(d) (5) $\lim_{\theta \to 0} \frac{\sin(3\theta)}{\theta}$

Solution: This equals

$$\lim_{\theta \to 0} 3 \frac{\sin(3\theta)}{3\theta} = 3 \cdot 1 = 3.$$

2. The graph of a function f with $f(-1) = -.2$ and $f(2) = -1$ is shown below.



(a) (10) What are $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$?

Solution: From the graph and the given information, $\lim_{x \to -1^-} f(x) = -.2$ and $\lim_{x \to -1^+} f(x) = 2$.

(b) (15) Find all x in $(-3, 5)$ where f is discontinuous. Give the types of each of the discontinuities

Solution: $f(x)$ has jump discontinuities at $x = -1$ and $x = 2$. It also has an infinite discontinuity (vertical asymptote) at $x = 3$. These are the only discontinuities.

(c) (10) Given that $f(x) = x + 3$ for $-1 < x < 0$ and $f(x) = 3 - x - \frac{x^3}{8}$ for $0 \le x < 2$, is f differentiable at $a = 0$? Why or why not?

Solution: The answer is *no*, f is not differentiable at 0. The easiest way to see this is that $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$ will agree with the derivative of $x + 3$ at $x = 0$, and equal 1. On the other hand $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$ will agree with the derivative of $3 - x - \frac{x^3}{8}$ at $x = 0$, which is -1. (This is the precise meaning of the apparent "corner" on the graph at $x = 0$.) Since the one-sided limits of the difference quotient of f are not the same, $f'(0)$ does not exist.

3. Do not use the "short-cut" differentiation rules from Chapter 3 in this question.

(a) (5) State the limit definition of the derivative $f'(x)$.

Solution: The derivative $f'(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists.

(b) Estimate the derivative of $f(x) = \sqrt{x+3}$ at $a = 6$ numerically by computing difference quotients of f with $h = \pm .1$, then $h = \pm .01$. Enter your values in the table below, and then state what your estimate of $f'(8)$ is.

Solution: Computing

$$\frac{\sqrt{6+h+3} - \sqrt{6+3}}{h} = \frac{\sqrt{9+h} - 3}{h}$$

for the given values of h we get the numbers in the table (rounded to 4 decimals):

h	$-.1$	$-.01$	$.01$	$.1$
difference quotient value	.1671	.1667	.1666	.1662

The limit is apparently around .1666.

(c) (10) Use the definition to compute the derivative function of $f(x) = \sqrt{x+3}$.

Solution: We compute the limit by multiplying top and bottom of the difference quotient by the conjugate radical:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{(\sqrt{x+h+3} + \sqrt{x+3})}{(\sqrt{x+h+3} + \sqrt{x+3})} \\ &= \lim_{h\to 0} \frac{(x+h+3) - (x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}} \\ &= \frac{1}{2\sqrt{x+3}}. \end{aligned}$$

(d) (5) Find the equation of the line tangent to the graph $y = \sqrt{x+3}$ at $a = 6$.

Solution: By the previous part, the slope of the tangent is $f'(6) = \frac{1}{6}$. The point $(6, f(6)) = (6, 3)$. The tangent line has equation:

$$y - 3 = \frac{1}{6}(x-6) \quad \text{or} \quad y = \frac{x}{6} + 2.$$

4. Use the short-cut rules to compute the following derivatives. You may use any correct method, but must show work for full credit.

(a) (5) $\dfrac{d}{dx}\left(\dfrac{3}{\sqrt{x}} - e^x + 3x\right)$

Solution: The function can also be written as

$$f(x) = 3x^{-1/2} - e^x + 3x$$

So the derivative is

$$f'(x) = \frac{-3}{2}x^{-3/2} - e^x + 3.$$

(b) (5) $\dfrac{d}{dv}\big((v^2 - 2v)(v^3 + 1)\big)$

Solution: Multiply out first. Our function is

$$g(v) = v^5 - 2v^4 + v^2 - 2v,$$

so the derivative is

$$g'(v) = 5v^4 - 8v^3 + 2v - 2.$$

(c) (5) $\dfrac{d}{dx}\left(\dfrac{2^e + e^2 - x^e}{4}\right)$

Solution: The first two terms in the sum are constants, hence have derivative equal to zero. The last term is a power so the derivative is $-\frac{e}{4}x^{e-1}$.