Holy Cross College, Fall Semester, 2016
MATH 135, Section 01, Final Exam Solutions
Friday, December 16, 8:00 AM

1. [5 points each] Circle the number of the graph (on next page) showing each of the following functions. Note that there is one "extra" graph that does not match any of these functions.

(a) $f(x) = e^{-x} + 2$ is Plot III

(b) $f(x) = x^3 - 4x$ is Plot V

(c) $f(x) = 2\cos(2\pi x)$ is Plot I

(d) $f(x) = \dfrac{1}{x^2 - 9}$ is Plot IV

(e) Give a formula of a function that matches the graph you did not circle. *Answer:* Plot II is the plot of $f(x) = \sin(x)$.

2. [20 points] One of the functions given in the following table is linear and the other is exponential. Find a formula for the linear one and place it in the appropriate box. In the box for the other one, write "Exponential."

x	1	2	3	4	5
$f(x)$	1.2	0.6	0.3	0.15	0.075
$g(x)$	-2.3	-0.6	1.1	2.8	4.5

Solution: $g(x)$ is the linear function $g(x) = 1.7(x-1) - 2.3 = 1.7x - 4$. $f(x)$ is the exponential function (whose formula is $f(x) = 2.4\left(\frac{1}{2}\right)^x$, but that was not asked for).

3.

(a) [15 points] The depth of water in a tank oscillates sinusoidally once every 4 hours according to $d(t) = 2\cos\left(\frac{\pi t}{2}\right) + 4$. Sketch the graph of the depth versus time.

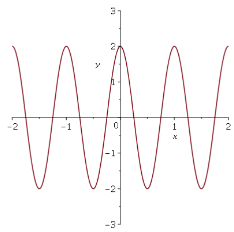
Solution: This is a cosine graph with amplitude 2, shifted up 4, and period 4. See plot on next page.

(b) [10 points] Find the average rate of change of the depth on the interval $[1, 1.1]$.

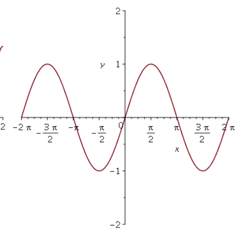
Solution: The average rate of change is

$$\frac{2\cos(1.1\pi/2) + 4 - (2\cos(\pi/2) + 4)}{1.1 - 1} \doteq -3.13$$

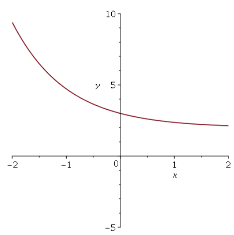
(If the units of d were feet and t was in hours, this would have the units of feet/hour, but that was not asked for.)



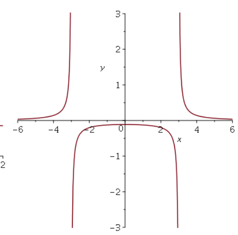
(a) Plot I



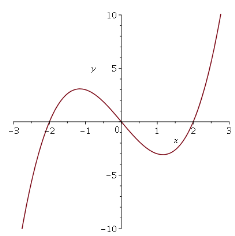
(b) Plot II



(c) Plot III



(d) Plot IV



(e) Plot V

Figure 1: Plots for problem 1.

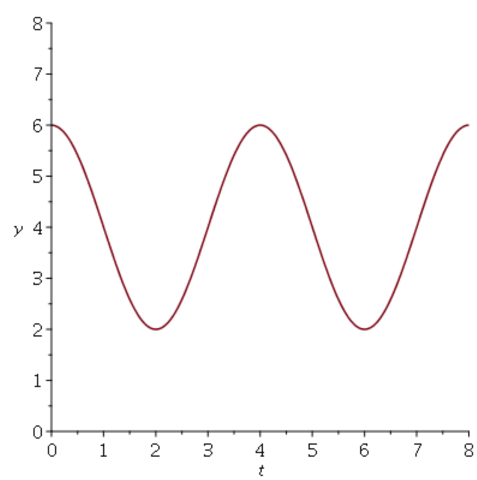


Figure 2: The plot $y = 2\cos\left(\frac{\pi t}{2}\right) + 4$ in question 3.

4. Compute the following limits [5 points each]. Any legal method is OK.

(a) $\lim_{x \to 2} \frac{x^3 + 2x}{x - 4}$

Solution: This rational function is continuous at $x = 2$, so the limit is equal to the value $12/(-2) = -6$.

(b) $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x^2 - 5x + 6}$

Soluiton: This is a 0/0 limit, so we factor the top and bottom, cancel and then evaluate:

$$= \lim_{x \to 2} \frac{(x-2)(x-5)}{(x-2)(x-3)} = \lim_{x \to 2} \frac{x-5}{x-3} = 3.$$

(c) $\lim_{x \to \infty} \frac{5x^2 - x + 21}{8x^2 - 9x + 1}$

Solution: Either divide the top and bottom by x^2 and take the limit, or use L'Hopital's Rule. The value is $\frac{5}{8}$.

5.

(a) [5 points] State the limit definition of the derivative:

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

(b) [10 points] Use the definition to compute $f'(x)$ for $f(x) = 3\sqrt{x+2}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{3\sqrt{x+h+2} - 3\sqrt{x+2}}{h} \\ &= \lim_{h \to 0} \frac{3\sqrt{x+h+2} - 3\sqrt{x+2}}{h} \cdot \frac{3\sqrt{x+h+2} + 3\sqrt{x+2}}{3\sqrt{x+h+2} + 3\sqrt{x+2}} \\ &= \lim_{h \to 0} \frac{9(x+2+h) - 9(x+2)}{h(3\sqrt{x+h+2} + 3\sqrt{x+2})} \\ &= \lim_{h \to 0} \frac{9}{3\sqrt{x+h+2} + 3\sqrt{x+2}} \\ &= \frac{3}{2\sqrt{x+2}}. \end{aligned}$$

(c) [10 points] Find the equation of the tangent line to the graph $y = 3\sqrt{x+2}$ at the point $(2, 6)$.

Solution: Since $f(2) = 3\sqrt{4} = 6$ and $f'(2) = \frac{3}{4}$, the tangent line is

$$y - 6 = \frac{3}{4}(x-2) \quad \text{or} \quad y = \frac{3}{4}x + \frac{9}{2}.$$

6. Compute the following derivatives using the derivative rules. You need not simplify. [5 points each]

(a) $f(t) = t^4 - \dfrac{1}{\sqrt[5]{t}} + e^t = t^4 - t^{-1/5} + e^t.$

Solution: $f'(t) = 4t^3 + \frac{1}{5}t^{-6/5} + e^t = 4t^3 + \frac{1}{5t^{6/5}} + e^t.$

(b) $g(x) = \dfrac{x^2 - 2}{\cos(x) + 1}$

Solution: By the quotient rule:

$$g'(x) = \frac{(\cos(x)+1)(2x) - (x^2-2)(-\sin(x))}{(\cos(x)+1)^2}.$$

(c) $h(z) = \ln(4z^2 + 2\tan^{-1}(z))$

Solution: By the chain rule

$$h'(z) = \frac{1}{4z^2 + 2\tan^{-1}(z)} \cdot \left(8z + \frac{1}{1+z^2}\right).$$

(d) Find $\frac{dy}{dx}$ if $5x^2y^2 - 2y^5 + x = 1$.

Solution: Using implicit differentiation:

$$10x^2y\frac{dy}{dx} + 10xy^2 - 10y^4\frac{dy}{dx} + 1 = 0,$$

so

$$\frac{dy}{dx} = \frac{-1 - 10xy^2}{10x^2y - 10y^4}.$$

7. All parts of this question refer to the functions defined by $f(x) = x^4 + 2ax^2$, where a is any fixed real number.

(a) [10 points] Assuming $a < 0$, find the *critical points* of f, and construct a sign diagram for $f'(x)$. Which of your critical points are local maxima and which are local minima?

Solution: Since $a < 0$, The derivative of f factors as

$$f'(x) = 4x^3 + 4ax = 4x(x^2 + a) = 4x(x + \sqrt{-a})(x - \sqrt{-a})$$

This is equal to zero at $x = 0, \pm\sqrt{-a}$. The derivative is negative on $(-\infty, -\sqrt{-a})$ and again on $(0, \sqrt{-a})$. It is positive on $(-\sqrt{-a}, 0)$ and $(\sqrt{-a}, +\infty)$. By the first derivative test, this says $x = \pm\sqrt{-a}$ are local minima and $x = 0$ is a local maximum.

(b) [10 points] Repeat part a, but assume now that $a > 0$.

Solution: We still have $f'(x) = 4x(x^2 + a)$, but when $a > 0$, the quadratic $x^2 + a = 0$ has no real roots, so $x = 0$ is the only critical point. Moreover, $x^2 + a > 0$ for all x, so the derivative is < 0 when $x < 0$ and > 0 when $x > 0$. The function has a local minimum at $x = 0$.

(c) [10 points] How many different *inflection points* does the graph $y = f(x)$ have if $a < 0$? Explain.

Solution: $f''(x) = 12x^2 + 4a$ is zero at $x = \pm\sqrt{-a/3}$ when $a < 0$. The second derivative changes sign at each of those points. So there are *two inflection points.*

8. [20 points] The radius and the height of a circular cone increase at a rate of 2 cm/sec. How fast is the volume of the cone increasing when $r = 10$ and $h = 20$?

Solution: From the volume formula for a cone, $V = \frac{\pi r^2 h}{3}$. Hence

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dh}{dt}\right).$$

Substituting $r = 10, h = 20$, $\frac{dr}{dt} = \frac{dh}{dt} = 2$, we get

$$\frac{dV}{dt} = \frac{\pi}{3}\left(10^2 \cdot 2 + 2 \cdot 10 \cdot 20 \cdot 2\right) = \frac{1000\pi}{3}$$

(cubic cm per second).

9. [20 points] A rectangular poster is to have total area 600 square inches, including blank 1 inch wide margins on all four sides of a central printed area. What overall dimensions will maximize the printed area?

Solution: Call the overall dimensions x, y. Then the dimensions of the printed area are $x - 2$ by $y - 2$ inches because of the blank margins. We have $xy = 600$ and we want to maximize

$$A = (x-2)(y-2) = (x-2)\left(\frac{600}{x} - 2\right) = 600 - \frac{1200}{x} - 2x + 4$$

We differentiate this and set it equal to zero to find the critical points:

$$A'(x) = \frac{1200}{x^2} - 2 = 0$$

so $x = \pm\sqrt{600}$. We discard the negative root because x is supposed to represent a length. So $x = 10\sqrt{6}$. Then $y = 10\sqrt{6}$ also. This is a maximum because $A''(x) = \frac{-2400}{x^3} < 0$ for all $x > 0$.