College of the Holy Cross, Spring 2014
Solutions for Math 136, Midterm Exam 3
May 2

I. Let R be the region bounded by $y = \sin(x)$, the x-axis, and $0 \le x \le \pi/3$.

A. [10 points] Write down (but *do not try to evaluate*) the integral that would compute the arc-length of the top edge of R.

Solution: The top is the graph $y = \sin(x)$, so $\frac{dy}{dx} = \cos(x)$, and the arclength is

$$\int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx = \int_0^{\pi/3} \sqrt{1 + \cos^2(x)}\; dx.$$

B. [15 points] A thin metal plate of constant density has the shape of R. Find the x-coordinate of the center of mass of the plate.

Solution: The x-coordinate of the center of mass is

$$\frac{\int_0^{\pi/3} x\sin(x)\; dx}{\int_0^{\pi/3} \sin(x)\; dx}$$

The denominator integral is simpler here:

$$\int_0^{\pi/3} \sin(x)\; dx = -\cos(x)|_0^{\pi/3} = 1 - \frac{1}{2} = \frac{1}{2}.$$

For the numerator, we need to integrate by parts (with $u = x$, $dv = \sin(x)\; dx$, so $du = dx$ and $v = -\cos(x)$):

$$\begin{aligned} \int_0^{\pi/3} x\sin(x)\; dx &= -x\cos(x)|_0^{\pi/3} + \int_0^{\pi/3} \cos(x)\; dx \\ &= -\frac{\pi}{6} + \sin(x)|_0^{\pi/3} \\ &= \frac{\sqrt{3}}{2} - \frac{\pi}{6}. \end{aligned}$$

So the x-coordinate of the center of mass is

$$\overline{x} = \sqrt{3} - \frac{\pi}{3} \doteq .6849$$

II. Both parts of this problem deal with the differential equation $\frac{dy}{dx} = xy$.

A. [15 points] Find the general solution $y(x)$ of the equation by separating variables and integrating.

Solution: This is separable:

$$\begin{aligned} \frac{dy}{dx} &= xy \\ \int \frac{dy}{y} &= \int x \, dx \\ \ln|y| &= \frac{x^2}{2} + c \\ y &= ke^{x^2/2} \quad \text{where} \quad k = \pm e^c. \end{aligned}$$

B. [5 points] Find the particular solution $y(x)$ satisfying the initial condition $y(0) = 4$ and compute the exact value of $y(2)$.

Solution: With $y(0) = 4$, we get $4 = ke^0$, so $k = 4$. The particular solution is $y = 4e^{x^2/2}$. With $x = 2$, we have $y = 4e^2 \doteq 29.56$.

III. [10 points] A drug is administered to a patient intravenously at a constant rate of 10mg per hour. The patient's body breaks down the drug and removes it from the bloodstream at a rate proportional to the amount present, with some proportionality constant k. Write a differential equation for the function $Q(t)$ = amount of the drug present (in mg) in the bloodstream at time t (in hours) that describes this situation. Note: You *do not need to solve* the equation.

Solution: The differential equation comes from analyzing both contributions to the change of the amount of drug in the bloodstream:

$$\frac{dQ}{dt} = kQ + 10.$$

Here $k < 0$ is the proportionality constant giving the rate at which the drug is broken down and removed from the bloodstream. The term $+10$ represents the constant intravenous dose of 10mg per hour.

IV.

A. [10 points] Does the geometric series $\sum_{n=0}^{\infty} \cdot \frac{2^n}{\pi^n}$ converge or diverge? If it is convergent, say why and find the sum; if it is not convergent say why not.

Solution: This is a geometric series with $a = 1$ and $r = \frac{2}{\pi}$. Since $|r| < 1$, it converges to the sum

$$\sum_{n=0}^{\infty} \frac{2^n}{\pi^n} = \frac{1}{1 - \frac{2}{\pi}} = \frac{\pi}{\pi - 2} \doteq 2.752$$

B. [10 points] Explain why the Integral Test can be applied to the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$ and use it to determine if the series converges or diverges.

Solution: The nth term of the series is $\frac{1}{n^2+4} = f(n)$, for the function $f(x) = \frac{1}{x^2+4}$. This function is continuous at all $x > 1$, it is decreasing (since $f'(x) = \frac{-2x}{(x^2+4)^2} < 0$ for all $x > 1$), and $\lim_{x\to\infty} \frac{1}{x^2+4} = 0$. Hence the Integral Test applies and

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2+4} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+4} \\ &= \lim_{b\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_1^b \\ &= \lim_{b\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{b}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) \\ &= \frac{\pi}{4} - \frac{\tan^{-1}\left(\frac{1}{2}\right)}{2}.\end{aligned}$$

Since the improper integral converges (is finite), the same is true for the infinite series.

V. All parts of this question refer to the power series

$$\sum_{n=1}^\infty \frac{(x-1)^n}{n3^n}.$$

A. [15 points] Use the Ratio Test to determine the radius of convergence.

Solution: Applying the Ratio Test,

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x-1)^n} \right| &= \lim_{n\to\infty} \frac{n}{3(n+1)}|x-1| \\ &= \frac{1}{3}|x-1|.\end{aligned}$$

For absolute convergence, we need $\frac{1}{3}|x-1| < 1$, so $|x-1| < 3$, or $-2 < x < 4$.

The series is centered at $a = 1$, so the radius of convergence is 3.

B. [10 points] Test convergence at the endpoints of the interval from part A to determine the interval of convergence. Explain your conclusions.

Solution: The inequality $|x-1| < 3$ is equivalent to $-3 < x-1 < 3$, or $-2 < x < 4$. At $x = 4$, we substitute and obtain

$$\sum_{n=1}^\infty \frac{1}{n}.$$

This is the harmonic series which *diverges.* At $x = -2$, we have

$$\sum_{n=1}^\infty \frac{(-1)^n}{n}.$$

The Alternating Series Test implies that this *converges.* So the interval of convergence is $[-2, 4)$, or all x with $-2 \le x < 4$.