College of the Holy Cross, Spring 2014
Math 136, section 1 – Solutions for Midterm Exam 2
Friday, March 28

I. For these problems, you must show *all work necessary* to justify your answers, but you may consult the portions of the table of integrals provided as needed. If you use a table entry, identify it by number.

A. (15) Integrate with the partial fraction method: $\displaystyle\int \frac{3x+1}{x^3+16x}\,dx$

Solution: The denominator factors as $x^3+16x = x(x^2+16)$, so the partial fractions will be

$$\frac{3x+1}{x^3+16x} = \frac{A}{x} + \frac{Bx+C}{x^2+16}$$

Clearing denominators and collecting powers of x,

$$3x+1 = (A+B)x^2 + Cx + 16A,$$

so

$$A+B = 0, C = 3, 16A = 1$$

Hence $A = \frac{1}{16}$, $B = \frac{-1}{16}$, and $C = 3$. Then we integrate as follows

$$\begin{aligned} \int \frac{3x+1}{x^3+16x}\,dx &= \frac{1}{16}\int \frac{dx}{x} - \frac{1}{16}\int \frac{x}{x^2+16}\,dx + 3\int \frac{1}{x^2+16} \\ &= \frac{1}{16}\ln|x| - \frac{1}{32}\ln|x^2+16| + \frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right) + C. \end{aligned}$$

(Explanation: The first two integrals are du/u forms. The last one can be done in several ways, including use of # 17 in the table of integrals with $u = x$ and $a = 4$.)

B. (15) Integrate via a trigonometric substitution: $\displaystyle\int x^3\sqrt{81-x^2}\,dx$

Solution: From the $\sqrt{a^2-u^2}$ form, we want $x = 9\sin\theta$, so $dx = 9\cos\theta\ d\theta$ and $\sqrt{81-x^2} = \sqrt{81(1-\sin^2\theta)} = 9\cos\theta$. Hence the integral becomes

$$\begin{aligned} \int x^3\sqrt{81-x^2}\,dx &= 9^5\int \sin^3\theta\cos^2\theta\ d\theta \\ &= 9^5\int \sin^2\theta\cos^2\theta\cdot\sin\theta\ d\theta \\ &= 9^5\int (1-\cos^2\theta)\cos^2\theta\sin\theta\ d\theta \\ &= 9^5\left(\int \cos^2\theta\sin\theta\ d\theta - \int \cos^4\theta\sin\theta\ d\theta\right) \\ &= 9^5\left(\frac{-\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right) + C, \end{aligned}$$

(since both of the last integrals are $u^n\ du$ forms). Converting back to x we have $\frac{x}{9} = \sin\theta$, so $\cos\theta = \frac{\sqrt{81-x^2}}{9}$ and the integral simplifies to

$$-27(81-x^2)^{3/2} + \frac{(81-x^2)^{5/2}}{5} + C.$$

(Note: The trig integral $\int \sin^3\theta \cos^2\theta\ d\theta$ can also be done via the reduction formula # 86 (use the second form to reduce the power of $\cos\theta$), and then # 67.)

II.

A. (5) Use midpoint Riemann sums with $n = 2$ to approximate $\int_1^2 \sqrt{1+\ln(x)}\ dx$.

Solution: With $n = 2$, $\Delta x = 1/2$ and we subdivide with one intermediate point at $3/2$. The midpoint of the first half is $5/4$, the midpoint of the second half is at $7/4$. The midpoint sum is

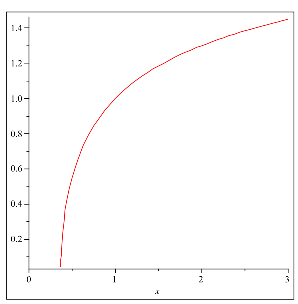
$$\frac{1}{2}\left(\sqrt{1+\ln(5/4)} + \sqrt{1+\ln(7/4)}\right) \doteq 1.1774$$

B. (5) Use the trapezoidal rule with $n = 2$ to approximate $\int_1^2 \sqrt{1+\ln(x)}\ dx$.

Solution: We compute

$$\frac{\Delta x}{2}\left(\sqrt{1+\ln(1)} + 2\sqrt{1+\ln(3/2)} + \sqrt{1+\ln(2)}\right) \doteq 1.1681.$$

C. (5) The function $y = \sqrt{1+\ln(x)}$ is plotted below.



Solution: Since the graph is concave down on this interval $[1, 2]$,

The midpoint approximation is an overestimate and

The trapezoidal rule approximation is an underestimate.

III. For each of the following improper integrals, set up and evaluate the appropriate limits to determine whether the integral converges. If so, find its value; if not, say "does not converge." (Credit will be given only for the correct limit calculation.)

A. (5) $\displaystyle\int_1^\infty e^{-x}\,dx$

This equals

$$\begin{aligned}\lim_{b\to\infty}\int_1^b e^{-x}\,dx &= \lim_{b\to\infty}\left(-e^{-x}\Big|_1^b\right.\\ &= \lim_{b\to\infty}\left(e^{-1}-e^{-b}\right)\\ &= e^{-1}-0\\ &= e^{-1}.\end{aligned}$$

Note that $\lim_{b\to\infty} e^{-b} = \lim_{b\to\infty}\frac{1}{e^b}$ does exist and equals *zero* because the numerator is constant but the denominator grows without bound as $b\to\infty$. Since the limit is finite, the integral converges.

B. (10) $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$

Solution: This integral is improper because the function $f(x)=\frac{1}{x^2}$ has a discontinuity at $x=0$, which is in the interval $[-1,1]$. When this happens we must consider

$$\lim_{b\to 0^-}\int_{-1}^b \frac{1}{x^2}\,dx + \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\,dx$$

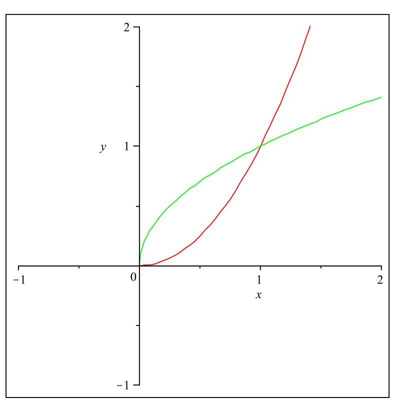
and both limits must exist for the integral to converge. But

$$\lim_{b\to 0^-}\int_{-1}^b \frac{1}{x^2}\,dx = \lim_{b\to 0^-}\left(\frac{-1}{x}\Big|_{-1}^b\right. = \lim_{b\to 0^-}\left(-1-\frac{1}{b}\right)\ d.n.e.$$

(The other integral also has an infinite limit as $a\to 0^+$.) This integral diverges as a result.

IV. The region R is bounded by the graphs $y=\sqrt{x}$ and $y=x^2$.

A. (5) Sketch the region R.



B. (5) Set up and compute an integral to find the area of R.

Solution: The area is

$$\int_0^1 \sqrt{x} - x^2 \, dx = \frac{2}{3}x^{3/2} - \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}.$$

C. (10) The region R is rotated about the x-axis to generate a solid. Set up and compute an integral to find its volume.

Solution: The cross-sections by planes $x = \text{const}$ are "washers" with outer radius $\sqrt{x}$ and inner radius x^2, so the volume is

$$V = \int_0^1 \pi(\sqrt{x})^2 - \pi(x^2)^2 \, dx = \pi\left(\frac{x}{2} - \frac{x^5}{5}\bigg|_0^1\right) = \frac{3\pi}{10}.$$

D. (10) A solid has the region R as base and cross-sections by planes perpendicular to the x-axis are isosceles right triangles with hypotenuse extending from the lower boundary to the upper boundary of the region. Set up and compute an integral to find the volume.

Solution: The hypotenuse of the triangle in the slice at x is $\sqrt{2}$ times the side, so the two equal sides of the triangle have length

$$\frac{\sqrt{x} - x^2}{\sqrt{2}}.$$

The area of the triangle is

$$\frac{1}{2} \cdot \left(\frac{\sqrt{x} - x^2}{\sqrt{2}}\right)^2 = \frac{(\sqrt{x} - x^2)^2}{4}.$$

By Cavalieri's Principle, the volume is

$$\frac{1}{4}\int_0^1 (\sqrt{x} - x^2)^2 \, dx = \frac{1}{4}\int_0^1 x - 2x^{5/2} + x^4 \, dx = \frac{1}{4}\left(\frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\bigg|_0^1\right) = \frac{9}{280}.$$

V. (10) Suppose that the region R defined by $0 \le y \le f(x)$ and $a \le x \le b$ has area A and lies above the x-axis. When R is rotated about the x-axis it sweeps out a solid with volume V_1. When R is rotated about the line $y = -k$, where $k > 0$, it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, A.

Solution: Since $-k < 0$, the cross-sections of the new solid are washers with outer radius $f(x) + k$ and inner radius k. The volume V_2 equals

$$V_2 = \int_a^b \pi(f(x) + k)^2 - \pi k^2 \, dx = \int_a^b \pi(f(x))^2 \, dx + 2k\pi \int_a^b f(x) \, dx$$

The first integral here computes the volume V_1 when R is rotated about the x-axis and the second computes the area of the region R. Hence this equals $V_1 + 2k\pi A$. We have

$$V_2 = V_1 + 2k\pi A.$$